Version of 14.4.26

The Prime Number Theorem

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The Prime Number Theorem, estimating the number of primes below a large number n, was one of the glories of nineteenth-century mathematics. Generations of young mathematicians have been taught that there is no easy proof. Slowly, however, ways have been found to proofs which are not exactly easy, but are short enough to be accessible. Here I will present a version of the proof in Zagier 97, itself derived from Newman 80, expanded to include those fragments of the theory of contour integration which are required.

1. Line integrals The first step is to get hold of some especially easy examples of integrals of complex functions along lines in the complex plane.

1A Vertical and horizontal line segments (a) Suppose that f is a bounded Borel measurable complex-valued function defined (at least) on a horizontal line segment from z_0 to z_1 where z_0, z_1 are complex numbers with $\mathcal{I}\mathrm{m}(z_0) = \mathcal{I}\mathrm{m}(z_1)$ and $\mathcal{R}\mathrm{e}(z_0) < \mathcal{R}\mathrm{e}(z_1)$. Then I define $\int_{[z_0,z_1]} f(z)dz$ to be $\int_{\mathcal{R}\mathrm{e}(z_0)}^{\mathcal{R}\mathrm{e}(z_1)} f(x + i\,\mathcal{I}\mathrm{m}(z_0))dx$, where this integral is with respect to ordinary one-dimensional Lebesgue measure on $\mathbb{R}$.

Similarly, for a vertical line segment from z_0 to z_1 where z_0, z_1 are complex numbers with $\mathcal{R}\mathrm{e}(z_0) = \mathcal{R}\mathrm{e}(z_1)$ and $\mathcal{I}\mathrm{m}(z_0) < \mathcal{I}\mathrm{m}(z_1)$, $\int_{[z_0,z_1]} f(z)dz$ will be $i\int_{\mathcal{I}\mathrm{m}(z_0)}^{\mathcal{I}\mathrm{m}(z_1)} f(\mathcal{R}\mathrm{e}(z_0) + iy)dy$.

(b)(i) Note that if z_0, z_1, z_2 are on the same vertical or horizontal line, with z_1 strictly between the other two, then $\int_{[z_0,z_1]} f(z)dz + \int_{[z_1,z_2]} f(z)dz = \int_{[z_0,z_2]} f(z)dz$ for any function f for which the third integral is defined.

(ii) We have a simple form for an integral $\int_{[z_0,z_1]} z\,dz$, which in both the cases considered in (a) above is $\frac{1}{2}(z_1^2 - z_0^2)$, as the formalism leads us to expect. **P** For a horizontal segment, we have

$$\begin{aligned}\int_{[z_0,z_1]} z\,dz &= \int_{\mathcal{R}\mathrm{e}(z_0)}^{\mathcal{R}\mathrm{e}(z_1)} x + i\,\mathcal{I}\mathrm{m}(z_0)dx \\ &= \frac{1}{2}((\mathcal{R}\mathrm{e}(z_1)^2 - (\mathcal{R}\mathrm{e}(z_0))^2) + i(\mathcal{R}\mathrm{e}(z_1) - \mathcal{R}\mathrm{e}(z_0))\,\mathcal{I}\mathrm{m}(z_0) \\ &= \frac{1}{2}((\mathcal{R}\mathrm{e}(z_1)^2 + 2i\,\mathcal{R}\mathrm{e}(z_1)\,\mathcal{I}\mathrm{m}(z_0) - \mathcal{I}\mathrm{m}(z_0)^2) \\ &\qquad - \frac{1}{2}(\mathcal{R}\mathrm{e}(z_0)^2 + 2i\,\mathcal{R}\mathrm{e}(z_0)\,\mathcal{I}\mathrm{m}(z_0) - \mathcal{I}\mathrm{m}(z_0)^2)) \\ &= \frac{1}{2}((\mathcal{R}\mathrm{e}(z_1) + i\,\mathcal{I}\mathrm{m}(z_1))^2 - (\mathcal{R}\mathrm{e}(z_0) + i\,\mathcal{I}\mathrm{m}(z_0))^2) = \frac{1}{2}(z_1^2 - z_0^2),\end{aligned}$$

and the calculation for vertical sections is similar. **Q**

(c) More generally, if f is any bounded Borel measurable complex-valued function and $[z_0, z_1]$ is either a vertical or horizontal line segment included in $\operatorname{dom} f$, then $|\int_{[z_0,z_1]} f(z)dz| \le \sup_{z\in[z_0,z_1]} |f(z)| \cdot |z_1 - z_0|$. **P** Translating this into one-dimensional integrals as in (a)-(b), this is the inequality $|\int_{x_0}^{x_1} f(x)dx| \le \int_{x_0}^{x_1} |f(x)|dx \le M(x_1 - x_0)$ where $M = \sup_{x\in[x_0,x_1]} |f(x)|$. **Q**

(d)(i) The formula in (c) echoes a similar formula when we have a function which is differentiable, as a complex function, along a line. Suppose that z_0 and z_1 are two points in $\mathbb{C}$, and write $[z_0, z_1]$ for the straight line segment from z_0 to z_1, that is, the set $\{\alpha z_0 + (1 - \alpha)z_1 : \alpha \in [0, 1]\}$. Now suppose that ϕ is a complex-valued function defined on and $[z_0, z_1]$ and differentiable along $[z_0, z_1]$ at each point of $[z_0, z_1]$, that is, $\phi'(w) = \lim_{z\in[z_0,z_1],z\to w} \frac{f(z)-f(w)}{z-w}$ in $\mathbb{C}$ for every $w \in [z_0, z_1]$. Then $|\phi(z_1) - \phi(z_0| \le \sup_{w\in[z_0,z_1]} |\phi'(w)| \cdot |z_1 - z_0|$. **P** Apply the one-dimensional real-variable version of the result to the function $\alpha \mapsto \mathcal{R}\mathrm{e}(v\phi(\alpha z_0 + (1 - \alpha)z_1)) : [0, 1] \to \mathbb{R}$ where $v \in \mathbb{C}$ and $|v| = 1$. **Q**

(ii) Next, suppose that z_0, z_1 are such that $\mathcal{I}\mathrm{m}(z_0) = \mathcal{I}\mathrm{m}(z_1)$ and $\mathcal{R}\mathrm{e}(z_0) < \mathcal{R}\mathrm{e}(z_1)$, and that ϕ is a bounded Borel measurable complex-valued function with domain including the line segment $[z_0, z_1]$. For $w \in \mathbb{C} \setminus [z_0, z_1]$ set $f(w) = \int_{[z_0,z_1]} \frac{1}{z-w}\phi(z)dz$. Then f is differentiable and $f'(w) = -\int_{[z_0,z_1]} \frac{1}{(z-w)^2}\phi(z)dz$ for every $w \in \mathbb{C} \setminus [z_0, z_1]$. **P** Note that as w does not belong to the compact set $[z_0, z_1]$, there is a $\delta > 0$ such that $|z - w| \ge 2\delta$ for every $z \in [z_0, z_1]$, so $z \mapsto \frac{1}{z-w}\phi(z)$ and $z \mapsto \frac{1}{(z-w)^2}\phi(z)$ are both bounded on $[z_0, z_1]$; as they are Borel measurable, the integrals are defined, for any particular w. Moreover, taking U to be the open disc $\{v : |v - w| < \delta\}$, $\sup\{|\frac{1}{(z-v)^2}\phi(z)| : v \in U,\ z \in [z_0, z_1]\} \le \frac{1}{\delta^2}\sup_{z\in[z_0,z_1]} |\phi(z)|$ is finite, so the argument for Corollary 123D in Fremlin 11, adapted to complex-valued integrands, applies to the function $(z, v) \mapsto \frac{1}{z-v}\phi(z) : [z_0, z_1] \times U \to \mathbb{C}$, and $f'(w) = -\int_{[z_0,z_1]} \frac{1}{(z-w)^2}\phi(z)dz$. **Q**

Similarly, this formula will be valid if $[z_0, z_1]$ is a vertical line segment. Furthermore, we can use the same argument to show that the second derivative $f''(w)$ is defined and equal to

$$-\int_{[z_0,z_1]} \frac{\partial}{\partial w}\frac{1}{(z-w)^2}\phi(z)dz = 2\int_{[z_0,z_1]} \frac{1}{(z-w)^3}\phi(z)dz$$

for every $w \in \mathbb{C} \setminus [z_0, z_1]$.

(e) Suppose that $[z_0, z_1]$ is a horizontal or vertical line segment as in (a)-(d), and $\langle\phi_n\rangle_{n\in\mathbb{N}}$ a uniformly bounded sequence of Borel measurable complex-valued functions, all with domain including $[z_0, z_1]$ and such that $\phi(z) = \lim_{n\to\infty} \phi_n(z)$ is defined in $\mathbb{C}$ for each $z \in [z_0, z_1]$. Setting $f(w) = \int_{[z_0,z_1]} \frac{1}{z-w}\phi(z)dz$ and $f_n(w) = \int_{[z_0,z_1]} \frac{1}{z-w}\phi_n(z)dz$ for $n \in \mathbb{N}$ and $w \in \mathbb{C}\setminus[z_0, z_1]$, $f(w) = \lim_{n\to\infty} f_n(w)$ and $f'(w) = \lim_{n\to\infty} f_n'(w)$ for every $w \in \mathbb{C} \setminus [z_0, z_1]$, by Lebesgue's Dominated Convergence Theorem.

1B Notation The theory here will involve integrals around closed curves; and the point of this note is that we can restrict attention to one very simple type.

(a) I will say that an **upright rectangle** is a set of the form $R = \{z : \alpha_0 \le \mathcal{R}\mathrm{e}(z) \le \alpha_1,\ \beta_0 \le \mathcal{I}\mathrm{m}(z) \le \beta_1\}$ where α_0, α_1, β_0, β_1 are real numbers, $\alpha_0 < \alpha_1$ and $\beta_0 < \beta_1$. Its boundary ∂R is the union of the line segments $[z_0, z_1]$, $[z_1, z_2]$, $[z_3, z_2]$ and $[z_0, z_3]$ where $z_0 = \alpha_0 + \beta_0 i$, $z_1 = \alpha_1 + \beta_0 i$, $z_2 = \alpha_1 + \beta_1 i$ and $z_3 = \alpha_0 + \beta_1 i$. If f is a bounded Borel measurable complex-valued function and $\partial R \subseteq \operatorname{dom} f$ then we shall need to look at the oriented sum $\int_{[z_0,z_1]} f(z)dz + \int_{[z_1,z_2]} f(z)dz - \int_{[z_3,z_2]} f(z)dz - \int_{[z_0,z_3]} f(z)dz$; I will call it $\int_{\partial R} f(z)dz$.

We can think of this as the integral along the oriented path $z_0 \to z_1 \to z_2 \to z_3 \to z_0$, where the leftwards and downwards segments are counted as negatives of rightwards and upwards segments $[z_3, z_2]$ and $[z_0, z_3]$. If you interpret the integral $\int_{[z_3,z_2]} f(z)dz$ as $\int_{z_3}^{z_2} f(z)dz$ this makes sense. But the whole point of the ideas to follow is that $\int_{[z_0,z_1]} + \int_{[z_1,z_2]}$ may be different from $\int_{[z_0,z_3]} + \int_{[z_3,z_2]}$. (See 1Ce below.) So we cannot safely think of either as being $\int_{z_0}^{z_2}$.

(b) If R, R' are upright rectangles which intersect in a shared edge, so that their union $R \cup R'$ is again an upright rectangle, then $\int_{\partial(R\cup R'} f(z)dz) = \int_{\partial R} f(z)dz + \int_{\partial R'} f(z)dz$ for every bounded Borel measurable complex-valued f defined on $\partial R \cup \partial R'$. **P** If R and R' share a vertical edge, so that they can be expressed as $\{z : \alpha_0 \le \mathcal{R}\mathrm{e}(z) \le \alpha_1,\ \beta_0 \le \mathcal{I}\mathrm{m}(z) \le \beta_1\}$ and $\{z : \alpha_1 \le \mathcal{R}\mathrm{e}(z) \le \alpha_2,\ \beta_0 \le \mathcal{I}\mathrm{m}(z) \le \beta_1\}$ where $\alpha_0 < \alpha_1 < \alpha_2$ and $\beta_0 < \beta_1$, then

$$\begin{aligned}\int_{\partial R} f(z)dz &+ \int_{\partial R'} f(z)dz \\ &= \int_{[z_0,z_1]} f(z)dz + \int_{[z_1,z_2]} f(z)dz - \int_{[z_3,z_2]} f(z)dz - \int_{[z_0,z_3]} f(z)dz \\ &\quad + \int_{[z_1,z_4]} f(z)dz + \int_{[z_4,z_5]} f(z)dz - \int_{[z_2,z_5]} f(z)dz - \int_{[z_1,z_2]} f(z)dz\end{aligned}$$

(where $z_0, \dots, z_3$ are as in (a), $z_4 = \alpha_2 + i\beta_0$ and $z_5 = \alpha_2 + i\beta_1$)

$$= \int_{[z_0,z_4]} f(z)dz + \int_{[z_4,z_5]} f(z)dz - \int_{[z_3,z_5]} f(z)dz - \int_{[z_0,z_3]} f(z)dz$$
$$= \int_{\partial(R\cup R'} f(z)dz$$

since $\int_{[z_0,z_1]} f(z)dz + \int_{[z_1,z_4]} f(z)dz = \int_{[z_0,z_4]} f(z)dz$ and $\int_{[z_3,z_2]} f(z)dz + \int_{[z_2,z_5]} f(z)dz = \int_{[z_3,z_5]} f(z)dz$ by 1A(b-i) above. **Q**

The same idea works if one of R, R' is stacked on top of the other.

1C Some special cases Take an upright rectangle R with vertices $z_0, \dots, z_3$ starting at bottom left and going anticlockwise, as in 1Bb.

(a) Corresponding to the special cases treated in 1Ab we find that if $f(z) = c_0 + c_1 z$ for $z \in \mathbb{C}$, where c_0, $c_1 \in \mathbb{C}$, than $\int_{\partial R} f(z)dz = 0$. **P**

$$\int_{\partial R} f(z)dz = \int_{[z_0,z_1]} f(z)dz + \int_{[z_1,z_2]} f(z)dz - \int_{[z_3,z_2]} f(z)dz - \int_{[z_0,z_3]} f(z)dz$$
$$= c_0 \int_{[z_0,z_1]} 1\,dz + c_1 \int_{[z_0,z_1]} z\,dz + c_0 \int_{[z_1,z_2]} 1\,dz + c_1 \int_{[z_1,z_2]} z\,dz$$
$$- c_0 \int_{[z_3,z_2]} 1\,dz - c_1 \int_{[z_0,z_3]} z\,dz - c_0 \int_{[z_0,z_3]} 1\,dz - c_1 \int_{[z_0,z_3]} z\,dz$$

(the formulae in 1Aa make it plain that integration is a linear operator)

$$= c_0(z_1 - z_0) + \frac{1}{2}c_1(z_1^2 - z_0^2) + c_0(z_2 - z_1) + \frac{1}{2}c_1(z_2^2 - z_1^2)$$
$$- c_0(z_2 - z_3) - \frac{1}{2}c_1(z_2^2 - z_3^2) - c_0(z_3 - z_0) - \frac{1}{2}c_1(z_3^2 - z_0^2)$$

(1Ab)

$$= 0.$$ **Q**

(b) If f is a bounded Borel measurable function with domain including ∂R, then $|\int_{\partial R} f(z)dz)| \le \operatorname{perim} R \cdot \sup_{z\in\partial R} |f(z)|$. **P** Apply 1Ac to each of the four sides of R. **Q**

(c) If ϕ is a bounded Borel measurable complex-valued function with domain including ∂R, then $f(w) = \int_{\partial R} \frac{1}{z-w}\phi(z)dz$ is defined for $w \in \mathbb{C} \setminus \partial R$, and f is differentiable, with $f'(w) = -\int_{\partial R} \frac{1}{(z-w)^2}\phi(z)dz$ for $w \in \mathbb{C} \setminus \partial R$. **P** Apply 1Ad to each side of R. **Q** In the same way, the second derivative $f''(w)$ is defined and equal to

$$-\int_{\partial R} \frac{\partial}{\partial w}\frac{1}{(z-w)^2}\phi(z)dz = 2\int_{\partial R} \frac{1}{(z-w)^3}\phi(z)dz$$

for every $w \in \mathbb{C} \setminus \partial R$.

(d) Suppose that R is an upright rectangle and $\langle\phi_n\rangle_{n\in\mathbb{N}}$ a uniformly bounded sequence of Borel measurable complex-valued functions with domain including ∂R, and such that $\phi(z) = \lim_{n\to\infty} \phi_n(z)$ is defined in $\mathbb{C}$ for each $z \in \partial R$. Setting $f(w) = \int_{\partial R} \frac{1}{z-w}\phi(z)dz$ and $f_n(w) = \int_{\partial R} \frac{1}{z-w}\phi_n(z)dz$ for $n \in \mathbb{N}$ and $w \in \mathbb{C} \setminus \partial R$, $f(w) = \lim_{n\to\infty} f_n(w)$ and $f'(w) = \lim_{n\to\infty} f_n'(w)$ for every $w \in \mathbb{C} \setminus \partial R$, as in 1Ae.

(e) If $R \subseteq \mathbb{C}$ is a square with centre w, and $f(z) = \frac{1}{z-w}$ for $z \in \mathbb{C} \setminus \{w\}$, then $\int_{\partial R} f(z)dz = 2\pi i$. **P** If $w = \gamma_0 + i\gamma_1$ and R has side length 2δ, then the integral $\int_{[z_0,z_1]} \frac{1}{z-w}dz$ becomes

$$\int_{\gamma_0-\delta}^{\gamma_0+\delta} \frac{1}{(x+i(\gamma_1-\delta))-(\gamma_0+i\gamma_1)}dx = \int_{-\delta}^{\delta} \frac{1}{x-i\delta}dx,$$

and similarly we have

$$\int_{[z_1,z_2]}\frac{1}{z-w}dz=i\int_{-\delta}^{\delta}\frac{1}{\delta+iy}dy,\qquad \int_{[z_3,z_2]}\frac{1}{z-w}dz=\int_{-\delta}^{\delta}\frac{1}{x+i\delta}dx,$$

$$\int_{[z_0,z_3]}\frac{1}{z-w}dz=i\int_{-\delta}^{\delta}\frac{1}{-\delta+iy}dy$$

,
so that

$$\int_{[z_0,z_1]}\frac{1}{z-w}dz-\int_{[z_3,z_2]}\frac{1}{z-w}dz=\int_{-\delta}^{\delta}\frac{1}{x-i\delta}-\frac{1}{x+i\delta}dx=\int_{-\delta}^{\delta}\frac{2i\delta}{x^2+\delta^2}dx$$
$$=2i[\arctan\frac{x}{\delta}]_{-\delta}^{\delta}=2i(\frac{\pi}{4}-(-\frac{\pi}{4}))=\pi i,$$

$$\int_{[z_1,z_2]}\frac{1}{z-w}dz-\int_{[z_0,z_3]}\frac{1}{z-w}dz=i\int_{-\delta}^{\delta}\frac{1}{\delta+iy}-\frac{1}{-\delta+iy}dy$$
$$=i\int_{-\delta}^{\delta}\frac{2\delta}{\delta^2+y^2}dy=\pi i$$

and $\int_{\partial R}f(z)dz=2\pi i$. **Q**

2. Holomorphic functions

2A I will use the phrase **holomorphic function** to mean a complex-valued function with domain an open subset of $\mathbb{C}$ which is differentiable, as a complex function, at every point of its domain.

If f and g are holomorphic functions then the composition fg is holomorphic, with domain $\{z : z\in\operatorname{dom}g, g(z)\in\operatorname{dom}f\}$; $f+g$ and $f\times g$ are holomorphic, with domain $\operatorname{dom}f\cap\operatorname{dom}g$; and f/g is holomorphic, with domain $\{z : z\in\operatorname{dom}f\cap\operatorname{dom}g,\ g(z)\neq 0\}$.

2B Cauchy's theorem, primitive form Let f be a holomorphic function and $R\subseteq\operatorname{dom}f$ an upright rectangle. Then $\int_{\partial R}f(z)dz=0$.

proof ? Suppose otherwise. Let $z_0,\dots,z_3$ be the vertices of R, taken anticlockwise from bottom left as in 1B and 1C, so that

$$\int_{\partial R}f(z)dz=\int_{[z_0,z_1]}f(z)\,dz+\int_{[z_1,z_2]}f(z)\,dz$$
$$-\int_{[z_0,z_3]}f(z)\,dz-\int_{[z_3,z_2]}f(z)\,dz.$$

Dissecting $R=R_0$ into four similar rectangles, with corners

$$(R_{00}):\quad z_0,\quad\frac{z_0+z_1}{2},\quad\frac{z_0+z_1+z_2+z_3}{4},\quad\frac{z_0+z_3}{2},$$
$$(R_{01}):\quad\frac{z_0+z_1}{2},\quad z_1,\quad\frac{z_1+z_2}{2},\quad\frac{z_0+z_1+z_2+z_3}{4},$$
$$(R_{02}):\quad\frac{z_0+z_1+z_2+z_3}{4},\quad\frac{z_1+z_2}{2},\quad z_2,\quad\frac{z_2+z_3}{2},$$
$$(R_{03}):\quad\frac{z_0+z_3}{2},\quad\frac{z_0+z_1+z_2+z_3}{4},\quad\frac{z_2+z_3}{2},\quad z_3$$

we see that

$$\int_{\partial R_0}f(z)dz=\int_{\partial(R_{00}\cup R_{01})}f(z)dz+\int_{\partial(R_{02}\cup R_{03})}f(z)dz$$
$$=\int_{\partial(R_{00})}f(z)dz+\int_{\partial(R_{01})}f(z)dz+\int_{\partial(R_{02})}f(z)dz+\int_{\partial(R_{03})}f(z)dz.$$

So there must be some $k \le 3$ such that $|\int_{\partial R_{0k}} f(z)dz| \ge \frac{1}{4}|\int_{\partial R_0} f(z)dz|$; setting $R_1 = R_{0k}$, R_1 is an upright rectangle, included in R_0, such that $|\int_{\partial R_1} f(z)dz| \ge \frac{1}{4}|\int_{\partial R_0} f(z)dz|$, while its perimeter $\operatorname{perim} R_1$ is just $\frac{1}{2}\operatorname{perim} R_0$.

Repeating this argument with R_1, we can find an upright rectangle $R_2 \subseteq R_1$ such that $|\int_{\partial R_1} f(z)dz| \ge \frac{1}{4}|\int_{\partial R_1} f(z)dz| \ge \frac{1}{16}|\int_{\partial R_0} f(z)dz|$ and $\operatorname{perim} R_2 = \frac{1}{4}\operatorname{perim} R_0$; continuing, we have a decreasing sequence $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots$ of rectangles with $|\int_{\partial R_n} f(z)dz| \ge 4^{-n}|\int_{\partial R} f(z)dz|$ and $\operatorname{perim} R_n = 2^{-n}\operatorname{perim} R$ for each n. Because R is compact, the intersection $\bigcap_{n\in\mathbb{N}} R_n$ will have a point $w \in R$, and f will be differentiable at w. Consider the function g defined by saying that $g(z) = f(z) - f(w) - (z-w)f'(w)$ for $z \in \operatorname{dom} g$. Then g is holomorphic and $g(w) = g'(w) = 0$, while $\int_{\partial R_n} g(z)dz = \int_{\partial R_n} f(z)dz$ for every n, by 1Ca.

Take $\epsilon > 0$. Let $\delta > 0$ be such that $w + z \in \operatorname{dom} g$ and $|g(w+z)| \le \epsilon|z|$ whenever $|z| \le \delta$, and take $n \ge 0$ such that $\operatorname{perim} R_n \le \delta$, so that $|g(z)| \le \epsilon|z-w| \le \epsilon \operatorname{perim} R_n$ for every $z \in R_n$. Now $|\int_{\partial R_n} g(z)dz| \le \epsilon(\operatorname{perim} R_n)^2$ by 1Cb, so

$$\begin{aligned} 4^{-n}|\int_{\partial R} f(z)dz| &\le |\int_{\partial R_n} f(z)dz| = |\int_{\partial R_n} g(z)dz| \\ &\le \epsilon(\operatorname{perim} R_n)^2 = 4^{-n}\epsilon(\operatorname{perim} R)^2 \end{aligned}$$

and $|\int_{\partial R} f(z)dz| \le \epsilon(\operatorname{perim} R)^2$. As ϵ is arbitrary, $\int_{\partial R} f(z)dz = 0$, which is what we wanted to know.

2C Cauchy's Integral Formula Let f be a holomorphic function and w a point in $\operatorname{dom} f$. If $R \subseteq \operatorname{dom} f$ is an upright rectangle and $w \in \operatorname{int} R = R \setminus \partial R$, then $f(w) = \frac{1}{2\pi i}\int_{\partial R} \frac{1}{z-w} f(z)dz$.

proof Set $g(z) = \frac{1}{z-w} f(z)$ for $z \in \operatorname{dom} f \setminus \{w\}$. Then g is holomorphic. Take $\epsilon > 0$. Let $\delta \in \left]0, \frac{1}{2}\epsilon\right]$ be such that the square R' with side length 2δ and centre w is included in $\operatorname{int} R$, and $|\frac{f(z)-f(w)}{z-w} - f'(w)| \le 1$ for $z \in R' \setminus \{w\}$. Projecting the left and right sides of R' to reach ∂R, R is divided into non-overlapping upright rectangles R', R_1, R_2, R_3, R_4, and for $1 \le k \le 4$ we have $R_k \subseteq \operatorname{dom} g$ so $\int_{\partial R_k} g(z)dz = 0$, by 2B. It follows that $\int_{\partial R} g(z)dz = \int_{\partial R'} g(z)dz$.

Set $h(z) = g(z) - \frac{f(w)}{z-w} - f'(w)$ for $z \in \operatorname{dom} f \setminus \{w\}$. Then h is holomorphic, and $\int_{\partial R'} h(z)dz = \int_{\partial R'} g(z)dz - 2\pi i f(w)$ by 1Ce and 1Ca. We have $|h(z)| \le 1$ for $z \in \partial R'$, so $|\int_{\partial R'} h(z)| \le \operatorname{perim} R' = 8\delta$. Accordingly

$$\begin{aligned} |f(w) - \frac{1}{2\pi i}\int_{\partial R} \frac{1}{z-w} f(z)dz| &= \frac{1}{2\pi}|2\pi i f(w) - \int_{\partial R'} g(z)dz| \\ &= \frac{1}{2\pi}|\int_{\partial R'} h(z)dz| \le \frac{8\delta}{2\pi} \le \epsilon. \end{aligned}$$

As ϵ is arbitrary, $f(w) = \frac{1}{2\pi i}\int_{\partial R} \frac{1}{z-w} f(z)dz$.

2D Corollary If f is a holomorphic function, its derivative is holomorphic.

proof Take any $w \in \operatorname{dom} f$. Then there is an upright rectangle R such that $w \in \operatorname{int} R$ and $R \subseteq \operatorname{dom} f$. Set $g(v) = \frac{1}{2\pi i}\int_{\partial R} \frac{1}{z-v} f(z)$ for $v \in \mathbb{C} \setminus \partial R'$. By 1Cc, g is twice differentiable. But 2C tells us that f and g agree on the open set $\operatorname{int} R$. So $f''(w) = g''(w)$ is defined. As w is arbitrary, f' is holomorphic.

2E Corollary Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of holomorphic functions, and $G \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n$ an open set such that (i) $\{f_n \restriction G : n \in \mathbb{N}\}$ is locally uniformly bounded (ii) $f(z) = \lim_{n\to\infty} f_n(z)$ is defined in $\mathbb{C}$ for every $z \in G$. Then f is holomorphic and $f'(z) = \lim_{n\to\infty} f'_n(z)$ for every $z \in G$.

proof (a) Take any $w \in G$. Then there is an upright rectangle $R \subseteq G$ such that $w \in \operatorname{int} R$. Because R is compact, $\{f_n \restriction R : n \in \mathbb{N}\}$ is uniformly bounded. By 2C, $f_n(w) = \int_{\partial R} \frac{1}{z-w} f_n(z)dz$ for $n \in \mathbb{N}$, so that

$$f(w)=\lim_{n\to\infty}f_n(w)=\lim_{n\to\infty}\int_{\partial R}\frac{1}{z-w}f_n(z)dz$$
$$=\int_{\partial R}\lim_{n\to\infty}\frac{1}{z-w}f_n(z)dz=\int_{\partial R}\frac{1}{z-w}f(z)dz$$

as in 1Cd. Moreover, 1Cd tells us that $f'(w)$ is defined and equal to $\lim_{n\to\infty}f_n'(w)$.

2F Taylor's theorem, primitive form Let f be a holomorphic function, $R\subseteq\operatorname{dom}f$ an upright rectangle and v a point in the interior of R. Set $\rho=\inf_{z\in\partial R}|z-v|$ and for $k\in\mathbb{N}$ set $a_k=\frac{1}{2\pi i}\int_{\partial R}\frac{1}{(z-v)^{k+1}}f(z)dz$. Then $f(w)=\sum_{k=0}^\infty a_k(w-v)^k$ whenever $|w-v|<\rho$.

proof For every $n\in\mathbb{N}$ and $z\in\partial R$,

$$\sum_{k=0}^{n-1}\frac{(w-v)^k}{(z-v)^k}=\frac{1-(\frac{w-v}{z-v})^n}{1-(\frac{w-v}{z-v})}=(1-(\frac{w-v}{z-v})^n)\frac{z-v}{z-w},$$

$$\textstyle\sum_{k=0}^{n-1}\frac{(w-v)^k}{(z-v)^{k+1}}=(1-(\frac{w-v}{z-v})^n)\frac{1}{z-w},$$

$$|\textstyle\sum_{k=0}^{n-1}\frac{(w-v)^k}{(z-v)^{k+1}}-\frac{1}{z-w}|=(\frac{|w-v|}{|z-v|})^n\frac{1}{|z-w|}\le(\frac{|w-v|}{\rho})^n\frac{1}{\rho-|w-v|}.$$

So

$$|\textstyle\sum_{k=0}^{n-1}\frac{(w-v)^k}{(z-v)^{k+1}}f(z)-\frac{1}{z-w}f(z)|\le M\frac{|w-v|^n}{\rho^n}\frac{1}{\rho-|w-v|}$$

where $M=\sup_{z\in\partial R}|f(z)|$, and $\sum_{k=0}^{n-1}\frac{(w-v)^k}{(z-v)^{k+1}}f(z)\to\frac{1}{z-w}f(z)$, uniformly for $z\in\partial R$, as $n\to\infty$. Integrating along ∂R,

$$f(w)=\frac{1}{2\pi i}\int_{\partial R}\frac{1}{z-w}f(z)dz=\frac{1}{2\pi i}\lim_{n\to\infty}\sum_{k=0}^{n-1}\int_{\partial R}\frac{(w-v)^k}{(z-v)^{k+1}}f(z)dz=\sum_{k=0}^{\infty}a_k(w-v)^k$$

as claimed.

2G Corollary Let f be a holomorphic function, and $w\in\operatorname{dom}f$ a zero of f. Then either there is a neigbourhood of w where f is constant and zero, or $\lim_{z\to w}\frac{(z-w)f'(z)}{f(z)}$ is defined and a strictly positive integer.

proof By 2F, there are a $\rho>0$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of complex numbers such that $f(z)=\sum_{n=0}^\infty a_n(z-w)^n$ whenever $|z-w|<3\rho$. If every a_n is zero, then we have the first alternative. Otherwise, there is a first m such that $a_m\ne 0$. As $a_0=f(w)=0$, m is greater than 0, while $f(z)=(z-w)^m\sum_{k=0}^\infty a_{k+m}(z-w)^k$ if $|z-w|<3\rho$. Since there is a v such that $2\rho<|v-w|<3\rho$, so that $\sum_{k=0}^\infty a_k(v-w)^k$ is defined and $M=\sup_{k\in\mathbb{N}}|a_k(v-w)^k|<\infty$, we see that $|a_{k+m}(z-w)^k|\le M\frac{|z-w|^k}{|v-w|^{k+m}}\le\frac{2^{-k}M}{|v-w|^m}$ for $k\ge 0$ and $\sum_{k=l}^\infty|a_{k+m}(z-w)^k|\le\frac{2^{-l+1}M}{|v-w|^m}$ whenever $l\ge 0$ and $|z-w|<\rho$.

Accordingly $g(z)=\sum_{k=0}^\infty a_{k+m}(z-w)^k$ is defined and the limit of the uniformly bounded sequence of its partial sums on $\{z:|z-w|<\rho\}$, and is holomorphic on $\{z:|z-w|<\rho\}$, while $g(w)=a_m$ is non-zero, and $f(z)=(z-w)^m g(z)$. So $f'(z)=m(z-w)^{m-1}g(z)+(z-w)^m g'(z)$ and $\frac{(z-w)f'(z)}{f(z)}=m+(z-w)\frac{g'(z)}{g(z)}\to m$ as $z\to w$.

2H Remarks (a) At a different level, we need to know that the exponential function is defined everywhere in $\mathbb{C}$ by the Maclaurin series $\exp(z)=\sum_{n=0}^\infty\frac{z^n}{n!}$ and that exp is holomorphic, with $\exp'=\exp$. Some of the proof can be shortened using the results above; but the basic theory of Maclaurin and Taylor series can be proved for complex numbers by the same methods as for real numbers. The point of 2B-2G above is that holomorphic complex functions from $\mathbb{C}$ to $\mathbb{C}$ are far more tightly circumscribed than differentiable real functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(b) If you have taken a first course in contour integration, everything here is likely to be familiar, though you may have been allowed to be bolder in some of the proofs. I have taken the trouble to spell this material out in order to make it plain that there is nothing genuinely difficult for anyone familiar with real ϵ-δ analysis. Of course I have omitted some of the most important ideas relating to integration along curves rather than simple polygons.

3. The Riemann zeta function

3A Definition For complex z such that $\mathcal{R}\mathrm{e}(z) > 1$, set

$$\zeta_0(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

For each $n \ge 1$, $z \mapsto \frac{1}{n^z} = \exp(-z \ln n)$ is holomorphic and $|\frac{1}{n^z}| = \frac{1}{n^{\mathcal{R}\mathrm{e}\, z}}$. So $\sum_{n=1}^{\infty} |\frac{1}{n^z}|$ is finite whenever $\mathcal{R}\mathrm{e}(z) > 1$, and the series is uniformly absolutely summable on $\{z : \mathcal{R}\mathrm{e}(z) > 1 + \delta\}$ for every $\delta > 0$. By 2E, ζ_0 is holomorphic.

Remark I write ζ_0 here, and ζ_1 in 3E below, to emphasize that these functions are restrictions of the true zeta function ζ to $\{z : \mathcal{R}\mathrm{e}(z) > 1\}$ and $\{z : \mathcal{R}\mathrm{e}(z) > 0\}$ respectively.

3B Theorem If $\mathcal{R}\mathrm{e}\, z > 1$, $\zeta_0(z) = \prod_{p \text{ prime}} \frac{1}{1-p^{-z}}$.

proof Enumerate the primes in increasing order as $\langle p_k \rangle_{k \ge 1}$. For any $n \ge 1$,

$$\prod_{k=1}^{n} \frac{1}{1-p_k^{-z}} = \prod_{k=1}^{n} \sum_{r=0}^{\infty} p_k^{-rz} = \sum_{r_1,\dots,r_n=0}^{\infty} p_1^{-r_1 z} \dots p_n^{-r_n z}$$
$$= \sum_{r_1,\dots,r_n=0}^{\infty} (p_1^{r_1} \dots p_n^{r_n})^{-z} = \sum_{m \in K_n} \frac{1}{m^z},$$

where K_n is the set of those $m \ge 1$ not divisible by any prime greater than p_n. Because the series $\langle \frac{1}{m^z} \rangle_{m \ge 1}$ is absolutely summable, and $\langle K_n \rangle_{n \ge 1}$ is non-decreasing with union $\mathbb{N} \setminus \{0\}$,

$$\zeta_0(z) = \lim_{n\to\infty} \sum_{m \in K_n} \frac{1}{m^z} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{1}{1-p_k^{-z}} = \prod_{p \text{ prime}} \frac{1}{1-p^{-z}}.$$

3C Corollary If $\mathcal{R}\mathrm{e}(z) > 1$, $\zeta_0(z) \ne 0$ and $\frac{\zeta_0'(z)}{\zeta_0(z)} = \sum_{p \text{ prime}} \frac{\ln p}{1-p^z}$.

proof For any particular z with $\mathcal{R}\mathrm{e}(z) > 1$, $|p^{-z}| < 1$ and $1 - p^{-z} \ne 0$ for every prime p, while $\sum_{p \text{ prime}} |p^{-z}| \le \sum_{n=1}^{\infty} n^{-\mathcal{R}\mathrm{e}\, z}$ is finite. So $\prod_{p \text{ prime}}(1 - p^{-z})$ is non-zero and its reciprocal $\prod_{p \text{ prime}} \frac{1}{1-p^{-z}}$ is non-zero. Moreover, if $\delta > 0$, then in the region $\{z : \mathcal{R}\mathrm{e}(z) \ge 1 + \delta\}$ the terms $1 - p^{-z}$ are bounded away from 0 and the sums $\sum_{p \text{ prime}} |p^{-z}|$ are bounded, so the products $\prod_{p \text{ prime}}(1 - p^{-z})$ are uniformly convergent.

Enumerating the primes as $\langle p_k \rangle_{k \ge 1}$ as in 3B, we see that

$$\frac{d}{dz} \prod_{k=1}^{m} \frac{1}{1-p_k^{-z}} = -\sum_{k=1}^{m} \frac{\frac{d}{dz}(1-p_k^{-z})}{1-p_k^{-z}} \prod_{k=1}^{m} \frac{1}{1-p_k^{-z}}$$
$$= -\sum_{k=1}^{m} \frac{p_k^{-z} \ln p_k}{1-p_k^{-z}} \prod_{k=1}^{m} \frac{1}{1-p_k^{-z}} = -\sum_{k=1}^{m} \frac{\ln p_k}{p_k^z - 1} \prod_{k=1}^{m} \frac{1}{1-p_k^{-z}}$$

whenever $m \ge 1$ and $\mathcal{R}\mathrm{e}(z) > 1$. Now 2E and 3B tell us not just that $z \mapsto \zeta_0(z) = \lim_{m\to\infty} \prod_{k=1}^m \frac{1}{1-p_k^{-z}}$ is holomorphic on $\{z : \mathcal{R}\mathrm{e}(z) > 1\}$ but also that $\zeta_0'(z) = \lim_{m\to\infty} \frac{d}{dz} \prod_{k=1}^m \frac{1}{1-p_k^{-z}}$. So

$$\frac{\zeta_0'(z)}{\zeta_0(z)} = -\lim_{m\to\infty} \sum_{k=1}^m \frac{\ln p}{p_k^z - 1} = \sum_{p \text{ prime}} \frac{\ln p}{1-p^z}$$

whenever $\mathcal{R}\mathrm{e}(z) > 1$.

3D Theorem There is a holomorphic function ϱ with domain $\{z : \mathcal{R}\mathrm{e}(z) > 0\}$ such that $\zeta_0(z) = \varrho(z) + \frac{1}{z-1}$ when $\mathcal{R}\mathrm{e}(z) > 1$.

proof If $n \ge 1$ and $\mathcal{R}\mathrm{e}(z) > 0$,

$$|\int_n^{n+1} (\frac{1}{n^z} - \frac{1}{x^z}) dx| = |\int_n^{n+1} \int_n^x \frac{z}{t^{z+1}} dt\, dx|$$

(the integrals here are with respect to Lebesgue measure)

$$\le \sup_{t\in[n,n+1]} |\frac{z}{t^{z+1}}| = \frac{|z|}{n^{\mathcal{R}\mathrm{e}(z)+1}}.$$

For each $n \ge 1$, $z \mapsto \int_n^{n+1} (\frac{1}{n^z} - \frac{1}{x^z}) dx : \mathbb{C} \to \mathbb{C}$ is holomorphic, just as in 1Ad above, while $\langle \int_n^{n+1} (\frac{1}{n^z} - \frac{1}{x^z}) dx \rangle_{n\ge1}$ is uniformly summable on $\{z : \mathcal{R}\mathrm{e}(z) \ge \delta,\ |z| \le \frac{1}{\delta}\}$ for every $\delta > 0$. So if we set $\varrho(z) = \sum_{n=1}^\infty \int_n^{n+1} (\frac{1}{n^z} - \frac{1}{x^z}) dx$ for $\mathcal{R}\mathrm{e}(z) > 0$, ϱ will be holomorphic, by 2E again.

Now observe that if $\mathcal{R}\mathrm{e}(z) > 1$,

$$\varrho(z) = \sum_{n=1}^\infty \frac{1}{n^z} - \int_1^\infty \frac{1}{x^z} dx = \zeta_0(z) + [\frac{1}{(z-1)x^{z-1}}]_1^\infty = \zeta_0(z) - \frac{1}{z-1},$$

so $\zeta_0(z) = \varrho(z) + \frac{1}{z-1}$.

3E Corollary Define ζ_1 by saying that $\operatorname{dom} \zeta_1 = \{z : \mathcal{R}\mathrm{e}(z) > 0,\ z \ne 1\}$ and $\zeta_1(z) = \varrho(z) + \frac{1}{z-1}$. Then ζ_1 is holomorphic and extends ζ_0.

3F Proposition Set $\varphi_0(z) = \sum_{p \text{ prime}} \frac{\ln p}{p^z}$ when $\mathcal{R}\mathrm{e}(z) > 1$, $\varkappa(z) = \sum_{p \text{ prime}} \frac{\ln p}{p^z(p^z-1)}$ when $\mathcal{R}\mathrm{e}(z) > \frac{1}{2}$, and $\varphi(z) = -\frac{\zeta_1'(z)}{\zeta_1(z)} - \varkappa(z)$ when $\mathcal{R}\mathrm{e}(z) > \frac{1}{2}$, $z \ne 1$ and $\zeta_1(z) \ne 0$. Then φ_0, $\varkappa$ and φ are holomorphic, and φ extends φ_0.

proof φ_0 is holomorphic because $\sum_{p \text{ prime}} |\frac{\ln p}{p^z}| < \infty$, just as for ζ_0 in 3A, while $\varkappa$ is well-defined and holomorphic because if $n \ge 4$ and $\gamma \ge \frac{1}{2}$ then $\frac{1}{2}n^\gamma \ge 1$ and $n^{2\gamma} - n^\gamma \ge \frac{1}{2}n^{2\gamma}$. So if $\delta > 0$ and $\mathcal{R}\mathrm{e}\, z \ge \frac{1}{2} + \delta$, then $|n^z(n^z - 1)| \ge |n^{2z}| - |n^z| \ge \frac{1}{2}n^{1+\delta}$ whenever $n \ge 4$, and

$$\sum_{n=4}^\infty |\frac{\ln n}{n^z(n^z-1)}| \le 2\sum_{n=4}^\infty \frac{\ln n}{n^{1+\delta}} < \infty.$$

Now

$$\varphi_0(z) + \varkappa(z) = \sum_{p \text{ prime}} \ln p (\frac{1}{p^z} + \frac{1}{p^z(p^z-1)})$$
$$= \sum_{p \text{ prime}} \frac{\ln p}{p^z - 1} = -\frac{\zeta_0'(z)}{\zeta_0(z)}$$

by 3C, so φ extends φ_0.

3G Remark Note that all the functions we have been examining here, from exp and $z \mapsto n^z$ and ζ_0 to φ_0, commute with complex conjugation, that is, $f(\bar{z})$ is defined and equal to $\overline{f(z)}$ whenever $z \in \operatorname{dom} f$.

3H Theorem $\zeta_1(z) \neq 0$ whenever $\mathcal{R}\mathrm{e}(z) \geq 1$ and $z \neq 1$.

proof (a) If $\mathcal{R}\mathrm{e}(z) > 1$ then $\zeta_1(z) = \zeta_0(z) \neq 0$, as observed in 3C.

(b) If $\mathcal{R}\mathrm{e}(w) = 1$ and $w \neq 1$, $\zeta_1(w+\epsilon) = \zeta_0(w+\epsilon) \neq 0$ for every $\epsilon > 0$, and ζ_1 is not constant zero on any neighbourhood of w. So $\lim_{z\to w}(z-w)\frac{\zeta_1'(z)}{\zeta_1(z)}$ is zero if $\zeta_1(w) \neq 0$, and otherwise is a strictly positive integer, by 2G. Also $\lim_{z\to w}(z-w)\varkappa(z) = 0 \cdot \varkappa(w) = 0$, so $\lim_{z\to w}(z-w)\varphi(z)$ is zero if $\zeta_1(w) \neq 0$, and otherwise is a strictly negative integer.

As for the case $w = 1$, we have $\zeta_0(z) = \varrho(z) + \frac{1}{z-1}$ and $\zeta_0'(z) = \varrho'(z) - \frac{1}{(z-1)^2}$ when $\mathcal{R}\mathrm{e}(z) > 1$, by 3D, so

$$\frac{\zeta_0'(z)}{\zeta_0(z)} = \frac{(z-1)^2\varrho'(z)-1}{(z-1)^2\varrho(z)+z-1}$$

when $\mathcal{R}\mathrm{e}\, z > 1$. and

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon\varphi(1+\epsilon) &= -\lim_{\epsilon\downarrow 0}\frac{\epsilon\zeta_0'(\epsilon)}{\zeta_0(\epsilon)} - \lim_{\epsilon\downarrow 0}\epsilon\varkappa(1+\epsilon)\\ &= -\lim_{\epsilon\downarrow 0}\frac{\epsilon^2\varrho'(1+\epsilon)-1}{\epsilon\varrho(1+\epsilon)+1} = 1.\end{aligned}$$

(c) ? If $\mathcal{R}\mathrm{e}\, z = 1$ and $z \neq 1$ and $\zeta_1(z) = 0$, then $z = 1+e^{i\alpha}$ for some $\alpha \neq 0$, and $\lim_{\epsilon\downarrow 0}\epsilon\varrho_0(1+e^{i\alpha}+\epsilon) = -k$ for some integer $k > 0$. Also $\lim_{\epsilon\downarrow 0}\epsilon\varrho_0(1+e^{2i\alpha}+\epsilon) = -l$ for some integer $l \geq 0$. Since φ_0 commutes with conjugation, $\lim_{\epsilon\downarrow 0}\varphi_0(e^{-i\alpha}+\epsilon) = -k$ and $\lim_{\epsilon\downarrow 0}\epsilon\varphi_0(e^{-2i\alpha}+\epsilon) = -l$. And we have just seen that $\lim_{\epsilon\downarrow 0}\epsilon\varphi(1+\epsilon) = 1$. But now consider

$$\begin{aligned}h(\epsilon) = \varphi_0(1+e^{-2i\alpha}+\epsilon) + 4\varphi_0(1+e^{-i\alpha}+\epsilon)&\\ + 6\varphi_0(1+\epsilon) + 4\varphi_0(1+e^{i\alpha}+\epsilon) + \varphi_0(1+e^{2i\alpha}+\epsilon)&\end{aligned}$$

for $\epsilon > 0$. Then $\lim_{\epsilon\downarrow 0}\epsilon h(\epsilon) = -l - 4k + 6 - 4k - l < 0$. On the other hand,

$$\begin{aligned}h(\epsilon) &= \sum_{p \text{ prime}}\frac{\ln p}{p^{1+\epsilon}}(p^{-2i\alpha} + 4p^{-i\alpha} + 6 + 4p^{i\alpha} + p^{2i\alpha})\\ &= \sum_{p \text{ prime}}\frac{\ln p}{p^{1+\epsilon}}(p^{-i\alpha/2} + p^{i\alpha/2})^4 > 0\end{aligned}$$

for $\epsilon > 0$. **X**

(d) Together with (a). this shows that $\zeta_1(z) \neq 0$ whenever $\mathcal{R}\mathrm{e}(z) \geq 1$ and $z \neq 1$.

4. The Prime Number Theorem

4A Proposition Setting $\vartheta(x) = \sum_{p \text{ prime}, p\leq x}\ln p$, $\vartheta(x) \leq 4x\ln 2$ for every real $x > 0$.

proof For any integer $n \geq 1$,

$$\begin{aligned}\vartheta(2n) - \vartheta(n) &= \sum_{p \text{ prime}, n<p\leq 2n}\ln p = \ln(\prod_{p \text{ prime}, n<p\leq 2n} p)\\ &\leq \ln(\frac{(2n)!}{(n!)^2}) \leq \ln(2^{2n}) = 2n\ln 2.\end{aligned}$$

In particular, $\vartheta(2^{k+1}) \leq \vartheta(2^k) + 2^{k+1}\ln 2$ for $k \geq 0$; inducing on k, $\vartheta(2^k) \leq 2\ln 2 \cdot 2^k$ for all k. Now for any $x > \frac{1}{2}$ there is a $k \in \mathbb{N}$ such that $x \leq 2^k < 2x$, so that

$$\vartheta(x) \leq \vartheta(2^k) \leq 2\ln 2 \cdot 2^k \leq 4x \ln 2.$$

4B Lemma If $\beta > 0$ and $\max(|\mathcal{R}\mathrm{e}(z)|, |\mathcal{I}\mathrm{m}(z)|) = \beta$ then $|1 + \frac{z^2}{\beta^2}| \leq \frac{2|z\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}$.

proof If z is of the form $x \pm i\beta$, then

$$|1 + \frac{z^2}{\beta^2}| = \frac{1}{\beta^2}|\beta^2 + x^2 \pm 2i\beta x - \beta^2| = \frac{1}{\beta^2}|x||x \pm 2i\beta|$$
$$\leq \frac{1}{\beta^2}|x||2x \pm 2i\beta| = \frac{2|x||z|}{\beta^2} = \frac{2|z\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}.$$

If z is of the form $iy \pm \beta$ where $|y| \leq \beta$, then

$$|1 + \frac{z^2}{\beta^2}| = \frac{1}{\beta^2}|2\beta^2 - y^2 \pm 2i\beta y| \leq \frac{1}{\beta^2}|2\beta^2 \pm 2i\beta y|$$
$$= \frac{1}{\beta^2}|\,\mathcal{R}\mathrm{e}(z)||2(\beta \pm iy)| = \frac{2|z\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}.$$

4C Lemma Suppose that $f : [0, \infty[\to \mathbb{C}$ is bounded and Borel measurable, and that g is a holomorphic function with domain including $\{z : \mathcal{R}\mathrm{e}\, z \geq 0\}$ and such that $g(z) = \int_0^\infty f(t)e^{-zt}dt$ whenever $\mathcal{R}\mathrm{e}(z) > 0$. Then $\lim_{\alpha\to\infty} \int_0^\alpha f(t)dt = g(0)$.

proof Take any $\epsilon > 0$.

(a) Set $M = \sup_{t\geq 0} |f(t)|$. Let $\beta > 0$ be such that $\frac{8M}{\beta} \leq \epsilon$, and let $\delta \in]0, \beta[$ be such that the upright rectangle $R = \{z : -\delta \leq \mathcal{R}\mathrm{e}(z) \leq \beta,\ |\mathcal{I}\mathrm{m}(z)| \leq \beta\}$ is included in $\mathrm{dom}\, g$. Set $K = \sup_{z\in\partial R} |g(z)|$ and let $m \geq 0$ be such that

$$\int_{[-\delta+\beta i,\beta i]} e^{m\,\mathcal{R}\mathrm{e}(z)}dz + \int_{[-\delta-\beta i,-\delta+\beta i]} e^{m\,\mathcal{R}\mathrm{e}(z)}dz + \int_{[-\delta-\beta i,-\beta i]} e^{m\,\mathcal{R}\mathrm{e}(z)}dz \leq \frac{\epsilon\delta}{3K};$$

the point here is that if $\mathcal{R}\mathrm{e}(z) < 0$ then $0 \leq e^{m\,\mathcal{R}\mathrm{e}(z)} \leq 1$ for all m and $\lim_{m\to\infty} e^{m\,\mathcal{R}\mathrm{e}(z)} \to 0$ as $m \to \infty$, so the integrals tend to 0 by 1Ae.

(b) Take any $\alpha \geq m$. Set $g_\alpha(z) = \int_0^\alpha f(t)e^{-zt}dt$ for $z \in \mathbb{C}$; then g_α is holomorphic, as in 1Ad above. Now set

$$h(z) = (g(z) - g_\alpha(z))(1 + \frac{z^2}{\beta^2})e^{\alpha z}$$

for $z \in \mathrm{dom}\, g$, so that h is holomorphic and defined on R and $h(0) = g(0) - g_\alpha(0)$. Then

$$g(0) - g_\alpha(0) = h(0) = \frac{1}{2\pi i}\int_{\partial R} \frac{h(z)}{z}$$

(by 2C)

$$= \frac{1}{2\pi i}\int_{\partial R} (g(z) - g_\alpha(z))(1 + \frac{z^2}{\beta^2})\frac{e^{\alpha z}}{z}dz.$$

(c) To estimate the integral, I break it into three parts.

(i) To the right, we have

$$\partial R \cap \{z : \mathcal{R}\mathrm{e}(z) \geq 0\} = [-\beta i, \beta(1-i)] \cup [\beta(1-i), \beta(i+1)] \cup [\beta i, \beta(i+1)]$$

giving a contribution

$$J_1 = \int_{[-\beta i,\beta(1-i)]} \frac{h(z)}{z} + \int_{[\beta(1-i),\beta(i+1)]} \frac{h(z)}{z} - \int_{[\beta i,\beta(i+1)]} \frac{h(z)}{z}.$$

In these integrals, we have (except at the endpoints $-\beta i$, βi)

$$|g(z)-g_\alpha(z)| = |\int_\alpha^\infty f(t)e^{-zt}dt| \le M\int_\alpha^\infty |e^{-zt}|dt$$
$$= M\int_\alpha^\infty e^{-t\,\mathcal{R}\mathrm{e}(z)}dt = \frac{Me^{-\alpha\,\mathcal{R}\mathrm{e}(z)}}{\mathcal{R}\mathrm{e}(z)}$$

and

$$|\frac{1}{z}e^{\alpha z}(1+\frac{z^2}{\beta^2})| \le \frac{1}{|z|}e^{\alpha\,\mathcal{R}\mathrm{e}(z)}\cdot\frac{2|z\,\mathcal{R}\mathrm{e}(z)|}{\beta^2} = e^{\alpha\,\mathcal{R}\mathrm{e}(z)}\cdot\frac{2|\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}$$

by 4B, so

$$|h(z)| \le \frac{Me^{-\alpha\,\mathcal{R}\mathrm{e}(z)}}{\mathcal{R}\mathrm{e}(z)}e^{\alpha\,\mathcal{R}\mathrm{e}(z)}\frac{2|\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}| = \frac{2M}{\beta^2};$$

as the sum of the lengths of the intervals is 4β, $|J_1| \le \frac{8M}{\beta} \le \epsilon$.

(ii) To the left, we have

$$\partial R\cap\{z:\mathcal{R}\mathrm{e}(z)\le 0\} = [-\delta+\beta i,\beta i]\cup[-\delta-\beta i,-\delta+\beta i]\cup[-\delta-\beta i,-\beta i].$$

Setting $h_2(z) = g(z)(1+\frac{z^2}{\beta^2})e^{\alpha z}$ for $z\in\operatorname{dom} g$ and $h_3(z) = g_\alpha(z)(1+\frac{z^2}{\beta^2})e^{\alpha z}$ for all $z\in\mathbb{C}$, we have a contribution J_2-J_3 to the integral, where

$$J_2 = -\int_{[-\delta+\beta i,\beta i]}\frac{h_2(z)}{z}dz - \int_{[-\delta-\beta i,-\delta+\beta i]}\frac{h_2(z)}{z}dz + \int_{[-\delta-\beta i,-\beta i]}\frac{h_2(z)}{z}dz,$$

$$J_3 = -\int_{[-\delta+\beta i,\beta i]}\frac{h_3(z)}{z}dz - \int_{[-\delta-\beta i,-\delta+\beta i]}\frac{h_3(z)}{z}dz + \int_{[-\delta-\beta i,-\beta i]}\frac{h_3(z)}{z}dz.$$

Now

$$|\frac{h_2(z)}{z}| \le \frac{3}{\delta}|g(z)|e^{\alpha\,\mathcal{R}\mathrm{e}\,z} \le \frac{3K}{\delta}e^{m\,\mathcal{R}\mathrm{e}(z)}$$

whenever $z\in\partial R$ and $\mathcal{R}\mathrm{e}\,z\le 0$, so $|J_2|\le\epsilon$, by the choice of m.

(iii) As for J_3, $h_3:\mathbb{C}\to\mathbb{C}$ is holomorphic and $z\mapsto\frac{h_3(z)}{z}:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ is holomorphic. So if R' is the upright rectangle $\{z:-\beta\le\mathcal{R}\mathrm{e}(z)\le-\delta,\ |\mathcal{I}\mathrm{m}(z)|\le\beta\}$ then $\int_{\partial R'}\frac{h_3(z)}{z}dz = 0$ and

$$J_3 = J_3 + \int_{\partial R'}\frac{h_3(z)}{z}dz$$
$$= -\int_{[-\beta+\beta i,\beta i]}\frac{h_3(z)}{z}dz - \int_{[-\beta-\beta i,-\beta+\beta i]}\frac{h_3(z)}{z}dz + \int_{[-\beta-\beta i,-\beta i]}\frac{h_3(z)}{z}dz.$$

Now if $\mathcal{R}\mathrm{e}(z)<0$,

$$|g_\alpha(z)| = |\int_0^\alpha f(t)e^{-zt}dt| \le M\int_0^\alpha |e^{-zt}|dt$$
$$= M\int_0^\alpha e^{-t\,\mathcal{R}\mathrm{e}(z)}dt = \frac{M(e^{-\alpha\,\mathcal{R}\mathrm{e}(z)}-1)}{|\,\mathcal{R}\mathrm{e}\,z|} \le \frac{Me^{-\alpha\,\mathcal{R}\mathrm{e}(z)}}{|\,\mathcal{R}\mathrm{e}\,z|}$$

while $|\frac{1}{z}e^{\alpha z}(1+\frac{z^2}{\beta^2})| = e^{\alpha\,\mathcal{R}\mathrm{e}(z)}\cdot\frac{2|\,\mathcal{R}\mathrm{e}(z)|}{\beta^2}$ as in (i), so $|h_3(z)|\le\frac{2M}{\beta^2}$ and $|J_3|\le\frac{8M}{\beta}\le\epsilon$, as before.

(d) Collecting these, we see that

$$|g(0)-g_\alpha(0)| = |\frac{1}{2\pi i}(J_1+J_2-J_3)| \le \frac{3\epsilon}{2\pi} \le \epsilon$$

for every $\alpha\ge\alpha_0$; as ϵ is arbitrary,

$$g(0) = \lim_{\alpha\to\infty} g_\alpha(0) = \lim_{\alpha\to\infty} \int_0^\alpha f(t)dt$$

as claimed.

4D Proposition $\lim_{\beta\to\infty} \int_1^\beta \frac{\vartheta(t)-t}{t^2} dt$ exists in $\mathbb{R}$.

proof (a) Observe that if $\mathcal{R}\mathrm{e}(z) > 1$ then

$$z\int_0^\infty e^{-zt}\vartheta(e^t)dt = z\int_0^\infty e^{-zt} \sum_{p \text{ prime}, p\le e^t} \ln p\, dt$$
$$= \sum_{p \text{ prime}} \int_{\ln p}^\infty z e^{-zt}\ln p\, dt = \sum_{p \text{ prime}} \frac{\ln p}{p^z} = \varphi_0(z) = \varphi(z)$$

by 3F. If we set $f(t) = \vartheta(e^t)e^{-t} - 1$ for $t \ge 0$, $-1 \le f(t) \le 4\ln 2 - 1$ for every $t \ge 0$, by 4A, so f is bounded and Borel measurable, and if $\mathcal{R}\mathrm{e}(z) > 0$, then

$$\int_0^\infty f(t)e^{-zt}dt = \int_0^\infty \vartheta(e^t)e^{-(z+1)t} - e^{-zt}dt = \frac{\varphi(z+1)}{z+1} - \frac{1}{z}.$$

(b) We saw in 3E that $\zeta_1(z) = \varrho(z) + \frac{1}{z-1}$ if $\mathcal{R}\mathrm{e}(z) > 0$ and $z \ne 1$. So $\zeta_1(z) \ne 0$ and

$$\frac{\zeta_1'(z)}{\zeta_1(z)} + \frac{1}{z-1} = (\rho'(z) - \frac{1}{(z-1)^2})/(\rho(z) + \frac{1}{z-1}) + \frac{1}{z-1} = \frac{(z-1)\rho'(z)+\rho(z)}{(z-1)\rho(z)+1}$$

if z is close to 1 and not equal to 1. But this means that if we set $h(z) = \frac{\zeta_1'(z)}{\zeta_1(z)} + \frac{1}{z-1} + \varkappa(z)$ when $\mathcal{R}\mathrm{e}(z) > \frac{1}{2}$ and $z \ne 1$ and $\zeta_1(z) \ne 0$, and $h(1) = \rho(1) + \varkappa(1)$, then $h'(1)$ is defined and equal to $1 + \kappa'(1)$, so h will be holomorphic, with domain $\{z : \mathcal{R}\mathrm{e}(z) > \frac{1}{2}, \zeta_1(z) \ne 0\}$, which is an open set including $\{z : \mathcal{R}\mathrm{e}(z) \ge 1\}$. 3F tells us also that $h(z) = \frac{1}{z-1} - \varphi(z)$ when $z \in \operatorname{dom} h$ and $z \ne 1$.

(c) Set

$$g(z) = -\frac{h(z+1)}{z+1} \text{ when } \mathcal{R}\mathrm{e}(z) > -\tfrac{1}{2} \text{ and } \zeta_1(z+1) \ne 0$$

so that g is holomorphic and its domain includes $\{z : \mathcal{R}\mathrm{e}(z) \ge 0\}$, while

$$g(z) = \frac{\varphi(z+1)}{z+1} - \frac{1}{z} = \int_0^\infty f(t)e^{-zt}dt$$

when $\mathcal{R}\mathrm{e}(z) > 0$.

(d) By Lemma 4C, $\lim_{\beta\to\infty} \int_0^\beta f(t)dt$ is defined in $\mathbb{R}$, that is,

$$\lim_{\beta\to\infty} \int_0^\beta \vartheta(e^t)e^{-t} - 1\, dt$$

is defined, that is,

$$\lim_{\beta\to\infty} \int_0^{\ln\beta} (\frac{\vartheta(t)}{t} - 1)\frac{1}{t}dt$$

is defined, and this is the required limit $\lim_{\beta\to\infty} \int_0^\beta \frac{\vartheta(t)-t}{t^2}dt$.

4E Corollary $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$.

proof (a) If $\lambda > 1$, $\int_1^\lambda bover\lambda - tt^2 dt > 0$, so there is a $\beta_0 > 0$ such that

$$\int_\beta^{\beta'} \frac{\vartheta(t)-t}{t^2}dt < \int_1^\lambda \frac{\lambda-t}{t^2}dt$$

whenever $\beta_0 \le \beta \le \beta'$. Now if $x > 0$ is such that $\vartheta(x) \ge \lambda x$, then

$$\int_x^{\lambda x}\frac{\vartheta(t)-t}{t^2}dt \ge \int_x^{\lambda x}\frac{\vartheta(x)-t}{t^2}dt \ge \int_x^{\lambda x}\frac{\lambda x-t}{t^2}dt = \int_1^{\lambda}\frac{\lambda-t}{t^2}dt$$

so $x \le \beta_0$. As λ is arbitrary, $\limsup_{x\to\infty}\frac{\vartheta(x)}{x} \le 1$.

(b) If $\lambda < 1$, then $\int_\lambda^1 \frac{\lambda-t}{t^2}dt < 0$, so there is a $\beta_0 > 0$ such that

$$\int_\beta^{\beta'}\frac{\vartheta(t)-t}{t^2}dt > \int_\lambda^1\frac{\lambda-t}{t^2}dt$$

whenever $\beta_0 \le \beta \le \beta'$. Now if $x > 0$ is such that $\vartheta(x) \le \lambda x$, then

$$\int_{\lambda x}^{x}\frac{\vartheta(t)-t}{t^2}dt \le \int_{\lambda x}^{x}\frac{\vartheta(x)-t}{t^2}dt \le \int_{\lambda x}^{x}\frac{\lambda x-t}{t^2}dt = \int_\lambda^1\frac{\lambda-t}{t^2}dt$$

and $\lambda x < \beta_0$. So $\liminf_{x\to\infty}\frac{\vartheta(x)}{x} \ge 1$. Together with (a), this shows that $\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1$.

4F The Prime Number Theorem Writing $\pi(x)$ for the number of primes less than or equal to x, $\lim_{x\to\infty}\frac{\pi(x)\ln x}{x} = 1$.

proof For any $x \ge 1$,

$$\vartheta(x) = \sum_{p\text{ prime},p\le x}\ln p \le \pi(x)\ln x$$

so $\liminf_{x\to\infty}\frac{\pi(x)\ln x}{x} \ge \lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1$.

On the other side, given $\epsilon \in \,]0,1[$, we have $\pi(x) = \pi(x^{1-\epsilon}) + \#(\{p : p \text{ is prime, } x^{1-\epsilon} < p \le x\})$ for every $x > 0$, so

$$\begin{aligned}\frac{\pi(x)\ln x}{x} &\le \frac{x^{1-\epsilon}\ln x}{x} + \sum_{p\text{ prime},x^{1-\epsilon}<p\le x}\frac{\ln x}{x}\\ &\le \frac{x^{1-\epsilon}\ln x}{x} + \sum_{p\text{ prime},x^{1-\epsilon}<p\le x}\frac{\ln p}{x(1-\epsilon)}\end{aligned}$$

(because if $x^{1-\epsilon} \le p$ then $(1-\epsilon)\ln x \le \ln p$)

$$\le \frac{\ln x}{x^\epsilon} + \frac{\vartheta(x)}{x(1-\epsilon)} \to 0 + \frac{1}{1-\epsilon}$$

as $x \to \infty$. So $\limsup_{x\to\infty}\frac{\pi(x)\ln x}{x} \le \frac{1}{1-\epsilon}$; as ϵ is arbitrary, $\limsup_{x\to\infty}\frac{\pi(x)\ln x}{x} \le 1$.

Putting these together,

$$\lim_{x\to\infty}\frac{\pi(x)\ln x}{x} \text{ is defined and equal to 1.}$$

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