POINTWISE COMPACT SETS OF BAIRE-MEASURABLE FUNCTIONS.

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Introduction

In this paper we study sets of real-valued functions defined on a topological space which lie within some class of "measurable" functions and satisfy some criterion of compactness or relative compactness in that class, which is always given the topology $\mathfrak{T}_p$ of "pointwise" or "simple" convergence. Our principal results seem to be the following:

1. Let X be a Polish space, $\boldsymbol{B}_1(X)$ the space of functions of the first Baire class. Then $\boldsymbol{B}_1(X)$ is angelic for $\mathfrak{T}_p$ (i.e., if $A \subseteq \boldsymbol{B}_1(X)$ is relatively countably compact, then A is relatively compact and every point in $\bar{A}$ is the limit of a sequence in A). Moreover, if $A \subseteq \boldsymbol{B}_1(X)$ is uniformly bounded and relatively compact, so is its convex hull $\Gamma(A)$ (Theorems 3F, 5E; our work is based on that of H. P. Rosenthal).

2. Let X be a Polish space, $\boldsymbol{B}(X)$ the space of all Borel real-valued functions on X. If A is a countable, relatively countably compact subset of $\boldsymbol{B}(X)$, then the closed convex hull of A in $\boldsymbol{B}(X)$ is compact and angelic (Theorem 4D and Proposition 5J).

3. Let X be a Polish space, $\hat{\boldsymbol{B}}(X)$ the space of real-valued functions on X with the Baire property. If A is a countable relatively countably compact set in $\hat{\boldsymbol{B}}(X)$, then A is relatively compact in $\hat{\boldsymbol{B}}(X)$, and for every $x \in \bar{A}$ there is a sequence in A converging to x except perhaps on a meager set (Theorem 6B).

4. On analyzing the lemmas required to prove these theorems, we find ourselves with striking results concerning sets of continuous functions which are relatively compact in some space of "measurable" functions. We set some of these out in Theorem 2F. In addition, several of our results apply, for one reason or another, to compact Hausdorff spaces as well as to Polish spaces.

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1. The First Baire Class and Related Spaces.

In this section we give definitions of the function spaces with which we shall be concerned, with some of the elementary relations between them.

Although in our view the importance of our results lies mainly in their application to Polish spaces, we find that our arguments often extend to other classes of space; in particular, many results can be transferred, with modifications, to compact Hausdorff spaces. Accordingly we shall give our basic definitions and lemmas in as much generality as seems convenient. The reader who is uninterested in "Čech-complete" and "K-analytic" spaces will find that he loses little by taking every underlying space to be either Polish or compact; in fact all the hardest arguments are already needed if $X=[0,1]$.

1A. Definitions. Let X be a topological space. The following is a list of the spaces of functions defined on X which will be used; it is complete except for $\boldsymbol{Ka}(X)$, defined in 4B below. The important ones to note immediately are (c) and (j).

(a) $\mathbf{R}^X$ is the space of all real-valued functions on X.

(b) $\boldsymbol{C}(X)$ is the space of continuous real-valued functions on X.

(c) $\boldsymbol{B}_1(X)$, the first Baire class, is the space of functions $x:X\to\mathbf{R}$ such that $x^{-1}[F]$ is G_δ in X for every closed set $F\subseteq\mathbf{R}$ (see Proposition 1E below).

(d) $\boldsymbol{B}_2(X)$, the second Baire class, is the space of functions $x:X\to\mathbf{R}$ such that $x^{-1}[F]$ is $F_{\sigma\delta}$ in X for every closed set $F\subseteq\mathbf{R}$. If X is metrizable, this is just the set of $\mathfrak{T}_p$-limits of sequences in $\boldsymbol{B}_1(X)$ [9, §31, Part VIII, Theorem 2].

(e) $\boldsymbol{B}(X)$ is the space of Borel-measurable real-valued functions on X.

(f) $\hat{\boldsymbol{B}}(X)$, the space of functions with the weak Baire property, is

$$\{x:x\in\mathbf{R}^X,\ \exists y\in\boldsymbol{B}(X)\text{ such that }\{t:x(t)\neq y(t)\}\text{ is meager}\}$$
$$=\{x:x\in\mathbf{R}^X,\ \exists\text{ comeager }Y\subseteq X\text{ such that }x|_Y\in\boldsymbol{C}(Y)\}$$

([9], §32; the restriction there to metric X is irrelevant for the moment).

(g) $\hat{\boldsymbol{B}}_r(X)$, the space of functions with the strong Baire property, is

$$\{x:x\in\mathbf{R}^X,\ x|_F\in\hat{\boldsymbol{B}}(F)\ \forall\text{ closed }F\subseteq X\}.$$

(h) $\boldsymbol{Ba}(X)$, the space of Baire measurable functions on X, is the σ-sublattice of $\mathbf{R}^X$ generated by $\boldsymbol{C}(X)$.

(i) $\boldsymbol{B}^{\S}(X)$ is the set of those functions $x: X \to \mathbf{R}$ such that, for any disjoint closed sets $E, F \subseteq \mathbf{R}$, $\operatorname{int} \overline{x^{-1}[E]} \cap \operatorname{int} \overline{x^{-1}[F]} = \varnothing$. (See Lemma 1C et seq.)

(j) $\boldsymbol{B}_r^{\S}(X) = \{x : x \in \mathbf{R}^X, x|_F \in \boldsymbol{B}^{\S}(F)\ \forall \text{ closed } F \subseteq X\}$.

(k) If μ is a measure on X, we shall write $\boldsymbol{M}_\mu(X)$ for the space of μ-measurable real-valued functions on X.

(l) If X is Hausdorff, $\boldsymbol{M}_r(X)$ will be the space of universally measurable functions on X, i.e., $\{x : x \in \boldsymbol{M}_\mu(X)$ for every Radon measure μ on $X\}$. (It will make no difference which conventions on the definition of Radon measures are followed. For definiteness we shall use those of [5], but [1, Chapter IX] or [13] is equally effective here.)

1B. Definition. A topological space is *Baire* if the intersection of any sequence of dense open sets is dense. It is *hereditarily Baire* if every closed subset is Baire in its subspace topology.

1C. Lemma. *Let X be a Baire space.*

(a) *For any* $x \in \mathbf{R}^X$, *the following are equivalent*:

  (i) $x \in \boldsymbol{B}^{\S}(X)$;

  (ii) *whenever* $\alpha < \beta$ *in* $\mathbf{R}$, *then* $\operatorname{int} \overline{S(\alpha)} \cap \operatorname{int} \overline{T(\beta)} = \varnothing$, *where* $S(\alpha) = \{t : x(t) \leqslant \alpha\}$, $T(\beta) = \{t : x(t) \geqslant \beta\}$;

  (iii) *whenever* $U \subseteq X$ *is a non-empty open set, and* $\varepsilon > 0$, *then there is a non-empty open set* $V \subseteq U$ *such that* $\operatorname{diam}(x[V]) \leqslant \varepsilon$;

  (iv) $(t : t \in X$, *x is continuous at* $t\}$ *is dense in X.*

(b) *Every upper or lower semicontinuous function on X belongs to* $\boldsymbol{B}^{\S}(X)$.

(c) $\boldsymbol{B}_1(X) \subseteq \boldsymbol{B}^{\S}(X)$.

(d) $\boldsymbol{B}^{\S}(X) \subseteq \hat{\boldsymbol{B}}(X)$.

(e) $\boldsymbol{B}^{\S}(X)$ *is a uniformly closed Riesz subspace* (= *vector sublattice*) *of* $\mathbf{R}^X$.

Proof. (a): (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iv). If x satisfies condition (ii), then set, for each $\alpha < \beta$ in $\mathbf{R}$,

$$G_{\alpha\beta} = \operatorname{int}\{t : x(t) < \beta\} \cup \operatorname{int}\{t : x(t) > \alpha\}.$$

Then $G_{\alpha\beta}$ is dense, because $G_{\alpha\beta} = \operatorname{int}[X \setminus T(\beta)] \cup \operatorname{int}[X \setminus S(\alpha)]$, so that $\overline{G}_{\alpha\beta} = X \setminus [\operatorname{int} \overline{S(\alpha)} \cap \operatorname{int} \overline{T(\beta)}] = X$.

Set $Y = \bigcap \{G_{\alpha\beta} : \alpha, \beta \in \mathbf{Q}, \alpha < \beta\}$. Then Y is dense, because X is Baire. We show that x is continuous at every point of Y. If $t_0 \in Y$ and $\varepsilon > 0$, then there exist $\alpha, \beta \in \mathbf{Q}$ such that $x(t_0) \leqslant \alpha < \beta \leqslant x(t_0) + \varepsilon$; now $t_0 \in G_{\alpha\beta}$ and $t_0 \notin \operatorname{int}\{t : x(t)$

$>\alpha\}$, so $t_0 \in \text{int}\{t : x(t) < \beta\} \subseteq \text{int}\{t : x(t) \leqslant x(t_0) + \varepsilon\}$. Similarly, $t_0 \in \text{int}\{t : x(t) \geqslant x(t_0) - \varepsilon\}$. As ε is arbitrary, x is continuous at t_0. Thus x satisfies condition (iv).

(iv)⇒(iii) is obvious; if x is continuous at $t \in U$, then there must be a neighborhood V of t such that $\text{diam}(x[V]) \leqslant \varepsilon$.

(iii)⇒(i). Suppose, if possible, otherwise: that x satisfies condition (iii) but does not belong to $\boldsymbol{B}^{\S}(X)$. Then there are disjoint closed sets $E, F \subseteq \mathbf{R}$ such that the intersection U of the interiors of the closures of their inverse images is not empty. We see that $U \cap x^{-1}[E]$ and $U \cap x^{-1}[F]$ are both dense in U. Now, using conditon (iii), we can choose inductively a sequence $\langle V_n \rangle_{n \in \mathbf{N}}$ of non-empty open sets in U such that $V_0 \subseteq U$, $V_{n+1} \subseteq V_n$ for every n, and $\text{diam}(x[V_n]) \leqslant \varepsilon$ for every $n \in \mathbf{N}$. Now every V_n must meet both $x^{-1}[E]$ and $x^{-1}[F]$; if we choose $s_n \in V_n \cap x^{-1}[E]$ and $t_n \in V_n \cap x^{-1}[F]$, we see that $\langle x(s_n) \rangle_{n \in \mathbf{N}}$ and $\langle x(t_n) \rangle_{n \in \mathbf{N}}$ are Cauchy sequences in $\mathbf{R}$ which must have a common limit belonging to $E \cap F$. But E and F are supposed to be disjoint. This contradiction shows that (iii)⇒(i).

(b): This follows immediately from (a) (ii). If we know that either $S(\alpha)$ or $T(\beta)$ is closed, then $\text{int}\,\overline{S(\alpha)} \cap \text{int}\,\overline{T(\beta)} = \varnothing$ because $S(\alpha) \cap T(\beta) = \varnothing$.

(c): If $x \in \boldsymbol{B}_1(X)$ and E, F are disjoint closed sets in $\mathbf{R}$, then for any open $U \subseteq X$ we have $x^{-1}[E] \cap U$ and $x^{-1}[F] \cap U$ G_δ sets in U. If $U = \text{int}\,\overline{x^{-1}[E]} \cap \text{int}\,\overline{x^{-1}[F]}$, then they are both dense; so their intersection is also dense. But their intersection is empty, so $U = \varnothing$. As E and F are arbitrary, $x \in \boldsymbol{B}^{\S}(X)$.

(d): $\boldsymbol{B}^{\S}(X) \subseteq \hat{\boldsymbol{B}}(X)$ by the argument of (a) (ii)⇒(iv) above: the set Y there is comeager and $x|_Y$ is surely continuous.

(e): This follows at once from the characterization (a) (iii).

1D. Corollary. *Let X be a hereditarily Baire space.*

(a) *If $x \in \mathbf{R}^X$, then $x \in \boldsymbol{B}_r^{\S}(X)$ iff, for every non-empty closed set $F \subseteq X$ and every $\alpha < \beta$ in $\mathbf{R}$, one of $\overline{F \cap S(\alpha)}$, $\overline{F \cap T(\beta)}$ is not equal to F.*

(b) *$\boldsymbol{B}_1(X) \subseteq \boldsymbol{B}_r^{\S}(X) \subseteq \hat{\boldsymbol{B}}_r(X)$, and every upper or lower semicontinuous function on X belongs to $\boldsymbol{B}_r^{\S}(X)$.*

(c) *$\boldsymbol{B}_r^{\S}(X)$ is a uniformly closed Riesz subspace of $\mathbf{R}^X$.*

Proof. (a): Suppose that $x \in \boldsymbol{B}_r^{\S}(X)$. If $F \subseteq X$ is closed, then $x|_F \in \boldsymbol{B}^{\S}(X)$, so that $\text{int}_F \overline{F \cap S(\alpha)} \cap \text{int}_F \overline{F \cap T(\beta)} = \varnothing$, where int_F denotes interiors taken in F, whenever $\alpha < \beta$; so if $F \neq \varnothing$, $\overline{F \cap S(\alpha)}$ and $\overline{F \cap T(\beta)}$ cannot both be equal to F.

On the other hand, if $x \notin \boldsymbol{B}_r^{\S}(X)$, then there exist a closed set $F \subseteq X$ and $\alpha < \beta$ such that $G = \text{int}_F \overline{F \cap S(\alpha)} \cap \text{int}_F \overline{F \cap T(\beta)} \neq \varnothing$. Set $H = \overline{G}$. Then every open set meeting H also meets G, and therefore meets both $G \cap S(\alpha)$ and

$G \cap T(\beta)$; so that $\varnothing \neq H = \overline{H \cap S(\alpha)} = \overline{H \cap T(\beta)}$, and the condition fails.

(b) and (c) follow at once from parts (b)–(e) of Lemma 1C.

1E. Proposition. *Let X be a complete metric space, $x \in \mathbf{R}^X$. Then the following are equivalent*:

(i) $x \in \boldsymbol{B}_1(X)$;
(ii) $x \in \boldsymbol{B}_r^{\S}(X)$;
(iii) *x is expressible as the $\mathfrak{T}_p$ limit of a sequence in $\boldsymbol{C}(X)$*;
(iv) $x|_K \in \boldsymbol{B}_1(K)$ *for every compact $K \subseteq X$.*

Proof. (a): The equivalence of (i)–(iii) is the famous theorem due essentially to Baire. Note that in fact (iii)⇒(i) in any topological space, and that (ii)⇒(i)⇒(iii) in any metric space [9, §31, Part X, Theorem 2; Part VIII, Theorem 7]; and (i)⇒(ii) is in Corollary 1D(b).

(b): Clearly (i)⇒(iv). For the reverse, suppose that $x \notin \boldsymbol{B}_1(X) = \boldsymbol{B}_r^{\S}(X)$. Then there is a non-empty closed set $F \subseteq X$ and $\alpha < \beta$ such that

$$S = \{t : t \in F, x(t) \leqslant \alpha\}, \qquad T = \{t : t \in F, x(t) \geqslant \beta\}$$

are both dense in F [Lemma 1D(a)]. We can accordingly choose an increasing sequence of finite sets $A_n \subseteq S \cup T$ such that (i) $A_0 \neq \varnothing$; (ii) $A_{n+1} \supseteq A_n$ for every $n \in \mathbf{N}$; (iii) if $t \in A_{n+1,}$ there is an $s \in A_n$ such that $\rho(t,s) \leqslant 2^{-n}$; (iv) if $s \in A_n$, there is a $t \in A_{n+1}$ such that $\rho(s,t) \leqslant 2^{-n}$ and $|x(s) - x(t)| \geqslant \beta - \alpha$. (Here ρ is any metric for which X is complete.) It is now easy to see that if $K = \overline{\bigcup_{n \in \mathbf{N}} A_n}$, then K is totally bounded for ρ and therefore compact, and $S \cap K$ and $T \cap K$ are both dense in K, so that $x|_K \notin \boldsymbol{B}_r^{\S}(K) = \boldsymbol{B}_1(K)$.

1F. Proposition. *Let X be a Hausdorff topological space, and $x \in \mathbf{R}^X$ such that $x|_K \in \boldsymbol{B}^{\S}(K)$ for every compact $K \subseteq X$. Then $x \in \boldsymbol{M}_r(X)$.*

Proof. It is enough to show that x is measurable for every Radon measure μ on X for which $\mu X < \infty$. Given such a measure, and $\varepsilon > 0$, let $\langle E_n \rangle_{n \in I}$, where I is some countable set, be a maximal disjoint family of measurable sets of non-zero measure on each of which the oscillation of x is $\leqslant \varepsilon$. Suppose, if possible, that $\mu(X \setminus \bigcup_{n \in I} E_n) > 0$. Let $K \subseteq X \setminus \bigcup_{n \in I} E_n$ be a compact set, of positive measure, which is supporting in the sense that $\mu(K \cap G) > 0$ if G is any open set meeting K. Then $x|_K \in B^{\S}(K)$, so there is a non-empty relatively open $V \subseteq K$ on which the oscillation of x is $\leqslant \varepsilon$ (Lemma 1C(a) above). But now, because K is supporting, $\mu V > 0$, and we should have added V to $\langle E_n \rangle_{n \in I}$. This contradiction shows that the E_n essentially cover X. So there is a measurable function which approximates x within ε almost everywhere. As ε is arbitrary, x is measurable.

Remarks. We shall not in this paper have occasion to study the spaces $\boldsymbol{B}^{\S}$ and $\boldsymbol{B}_r^{\S}$ on topological spaces which are not Baire. One of the reasons that we think it is nevertheless worthwhile giving the general definition is that the greater part of the results in [12] can be regarded as theorems about $\boldsymbol{B}_r^{\S}(X)$, where X is separable and metrizable. (The "Discontinuity Criterion" of [12] is clearly intimately related to the definition we give of $\boldsymbol{B}_r^{\S}$.) The identification $\boldsymbol{B}_1(X) = \boldsymbol{B}_r^{\S}(X)$, which depends on X being complete [for instance, $\boldsymbol{B}_1(\mathbf{Q}) = \mathbf{R}^{\mathbf{Q}} \neq \boldsymbol{B}_r^{\S}(\mathbf{Q})$], is largely irrelevant.

2. Relatively Compact Sets in $C(X)$.

This section is the foundation of our work. Our aim here is to show that, for suitable spaces X (roughly speaking, those with an adequate supply of compact subsets), a pointwise bounded collection A of continuous functions on X either has all its adherent points in $\mathbf{R}^X$ "regular" in various senses, or has adherent points which are exceedingly irregular. In §4 we shall explain the techniques which enable us to apply these results to (countable) families of functions which are Borel measurable rather than continuous.

The pivot of our argument is condition (vi) of Theorem 2F. Together with the combinatorial lemma 2B (which is a kind of topological version of Theorem 2 of [11]), it is this condition which enables us to transfer our problems to the Cantor space $\mathcal{P}\mathbf{N}$, and to use the fact that a non-principal ultrafilter on $\mathbf{N}$ is an exceptionally irregular set (Lemma 2D).

2A. Čech-Complete Spaces. In order to apply our arguments simultaneously to compact Hausdorff spaces and to complete metric spaces, we employ the following definition. A topological space is *Čech-complete* if it is expressible as a G_δ subset of a compact Hausdorff space. The following are well known and easy to prove:

(a) any locally compact Hausdorff space is Čech-complete (being open in its one-point compactification);
(b) any complete metric space is Čech-complete (being G_δ in its Stone-Čech compactification);
(c) a closed subspace of a Čech-complete space is Čech-complete;
(d) a G_δ subspace of a Čech-complete space is Čech-complete;
(e) a Čech-complete space is hereditarily Baire (because a dense G_δ set in a Baire space is Baire).

(See [4], Chapter 3, §8, and Chapter 4, §3, Theorem 11.)

2B. Lemma. *Let X be a Čech-complete space and $\mathcal{A}$ a family of ordered pairs of open sets in X. Suppose that there is a non-empty set $Y \subseteq X$ such that $\mathcal{A}$ is "weakly dense" over Y in the sense that, whenever $E_0, \ldots, E_n$ is a finite string of open sets in X all meeting Y, there is a pair $(G,H) \in \mathcal{A}$ such that $G \cap E_i \cap Y \neq \varnothing$, $H \cap E_i \cap Y \neq \varnothing$ for every $i \leqslant n$. Then there is a sequence $\langle (G_n, H_n) \rangle_{n \in \mathbf{N}}$ in $\mathcal{A}$ and a compact set $K \subseteq X$ such that $\langle (G_n, H_n) \rangle_{n \in \mathbf{N}}$ is "completely independent" over K, i.e.*

$$K \cap \bigcap_{n \in I} G_n \cap \bigcap_{n \in \mathbf{N} \setminus I} H_n \neq \varnothing \qquad \forall I \subseteq \mathbf{N}.$$

Remark. Here, and later, $\bigcap_{n \in \varnothing} G_n$ can be interpreted as the universal class.

Proof. We begin by expressing X as $\bigcap_{n \in \mathbf{N}} X_n$, where each X_n is an open set in a compact Hausdorff space Z. All our topological operations from now on will be expressed in Z.

Let

$$\mathcal{B} = \{(G,H): G,H \subseteq Z \text{ are open, } (G \cap X, H \cap X) \in \mathcal{A}\}.$$

Observe that $\mathcal{B}$ is weakly dense over Y in the same sense that $\mathcal{A}$ is. We construct a sequence $\langle (G_n, H_n) \rangle_{n \in \mathbf{N}}$ in $\mathcal{B}$ and open sets C_{PQ} in Z to have the following properties:

(i) C_{PQ} is defined whenever (P,Q) is a partition of $\{0, \ldots, n\}$ for some $n \in \mathbf{N}$, and in this case C_{PQ} is a non-empty open set in Z, meeting Y, with $\overline{C}_{PQ} \subseteq X_n \cap \bigcap_{n \in P} G_n \cap \bigcap_{n \in Q} H_n$;
(ii) if $P \subseteq P'$ and $Q \subseteq Q'$, then $C_{P'Q'} \subseteq C_{PQ}$.

The construction proceeds as follows. As Y is non-empty, the hypothesis tells us that there is a $(G_0, H_0) \in \mathcal{B}$ such that $G_0 \cap Y \neq \varnothing$, $H_0 \cap Y \neq \varnothing$; choose non-empty open sets $C_{\{0\},\varnothing}$ and $C_{\varnothing,\{0\}}$ in Z such that both meet Y and their closures are included in $G_0 \cap X_0$, $H_0 \cap X_0$ respectively. For the inductive step, suppose that G_i and H_i have been chosen for $i \leqslant n$ and that C_{PQ} has been chosen for each partition (P,Q) of $\{0, \ldots, n\}$. By the inductive hypotheses, every C_{PQ} is an open set in Z meeting Y; as $\mathcal{B}$ is weakly dense over Y, there is a $(G_{n+1}, H_{n+1}) \in \mathcal{B}$ such that

$$G_{n+1} \cap C_{PQ} \cap Y \neq \varnothing, \qquad H_{n+1} \cap C_{PQ} \cap Y \neq \varnothing$$

for every partition (P,Q) of $\{0, \ldots, n\}$. Now, for each partition (P,Q) of $\{0, \ldots, n\}$, choose $C_{P \cup \{n+1\}, Q}$ and $C_{P, Q \cup \{n+1\}}$ to be open sets, meeting Y, included in C_{PQ}, and with closures included in $G_{n+1}\cap X_{n+1}, H_{n+1}\cap X_{n+1}$ respectively. Thus the induction continues.

Now set

$$K = \bigcap_{n\in\mathbf{N}} \bigcup \left\{ \bar{C}_{PQ} : (P,Q) \text{ a partition of } \{0,\ldots,n\} \right\}.$$

As K is a closed set in Z, it is compact. As $\bar{C}_{P,Q} \subseteq X_n$ whenever (P,Q) is a partition of $\{0,\ldots,n\}$, $K\subseteq X$. To see that $\langle (G_n,H_n)\rangle_{n\in\mathbf{N}}$ is completely independent over K, let I be any subset of $\mathbf{N}$. Then $\langle \bar{C}_{P_n,Q_n}\rangle_{n\in\mathbf{N}}$ is a decreasing sequence of closed sets in Z, where

$$P_n = \{i : i\leqslant n, i\in I\}, \qquad Q_n = \{i : i \leqslant n, i\notin I\};$$

as Z is compact,

$$\varnothing \neq \bigcap_{n\in\mathbf{N}} \bar{C}_{P_nQ_n} \subseteq K\cap \bigcap_{n\in I} G_n \cap \bigcap_{n\in\mathbf{N}\setminus I} H_n.$$

It follows at once that $\langle (G_n\cap X, H_n\cap X)\rangle_{n\in\mathbf{N}}$ is a sequence in $\mathcal{Q}$ which is completely independent over K, as required.

2C. Lemma. *Suppose that K,L are compact Hausdorff spaces, and that $\varphi : K\to L$ is a continuous surjection such that $\varphi[F]\neq L$ for any closed set $F\subset K$. Then $\varphi[A]$ is meager in L for every meager set $A\subseteq K$. Consequently, if $y\in\mathbf{R}^L$, then $y\in\hat{\mathbf{B}}(L)$ iff $y\varphi\in\hat{\mathbf{B}}(K)$.*

Proof. (a) Suppose, if possible, that $H\subseteq K$ is a nowhere dense closed set such that $\operatorname{int}\varphi[H]\neq\varnothing$ in L. Set $G=\varphi^{-1}[\operatorname{int}\varphi[H]]$; as φ is a continuous surjection, G is a non-empty open set in K. As H is nowhere dense, $G\nsubseteq H$, and $F=(K\setminus G)\cup H\subset K$. However, $\varphi[G]\subseteq\varphi[H]$, so

$$\varphi[F]\supseteq \big(\varphi[K]\setminus\varphi[G]\big)\cup\varphi[H] = L,$$

contradicting our hypothesis on φ.

It follows at once that $\varphi[A]$ is meager in L whenever A is meager in K.

(b) Now, if $y\in\hat{\mathbf{B}}(L)$, there is a dense G_δ set $B\subseteq L$ such that $y|_B$ is continuous. We know that

$$\varphi\Big[\overline{\varphi^{-1}[B]}\Big]\supseteq \bar{B} = L$$

because K is compact, so $\varphi^{-1}[B]$ must be dense in K; of course it is a G_δ set. The restriction of $y\varphi$ to $\varphi^{-1}[B]$ is continuous, so $y\varphi\in\hat{\mathbf{B}}(K)$. On the other hand, suppose that $y \in \mathbf{R}^L$ and that $y\varphi \in \hat{\boldsymbol{B}}(K)$. Let $A \subseteq K$ be a meager set such that $y\varphi|_{K\setminus A}$ is continuous. Then for any closed set $F \subseteq \mathbf{R}$,

$$\{u : y(u) \in F\} = \varphi[E],$$

where $E = (y\varphi)^{-1}[F]$, so that $\bar{E}\setminus A = E \setminus A$; accordingly

$$\overline{\varphi[E]} \setminus \varphi[E] = \varphi[\bar{E}] \setminus \varphi[E] \subseteq \varphi[\bar{E}\setminus E] \subseteq \varphi[A],$$

and $\varphi[E]\setminus\varphi[A]$ is relatively closed in $L\setminus\varphi[A]$. This shows that $y|_{L\setminus\varphi[A]}$ is continuous. But we know that $\varphi[A]$ is meager, so that $y \in \hat{\boldsymbol{B}}(L)$, as required.

2D. Lemma. *Let K be the Cantor set* $\mathscr{P}\mathbf{N}$, *with its usual compact metric topology and its usual measure* (*the normalized Haar measure if we identify K with* $(\mathbf{Z}_2)^{\mathbf{N}}$). *If* $\mathfrak{F}$ *is any non-principal ultrafilter on* $\mathbf{N}$, *and E is a Souslin set* (*i.e., the continuous image of a Polish space*) *which is a subset of either* $\mathfrak{F}$ *or* $K\setminus\mathfrak{F}$, *then E is meager and of zero measure. In particular,* $\mathfrak{F}$ *itself is not measurable and does not have the Baire property.*

Proof. (a) If $E \subseteq \mathfrak{F}$ is Souslin, then E has the Baire property [9, §11, Part VII; §39, Part II]; let $G \subseteq K$ be an open set such that $E \triangle G$ is meager.

Suppose, if possible, that $G \neq \varnothing$. Consider the homeomorphisms $\vartheta_I : K \to K$ given by

$$\vartheta_I(A) = A \triangle I \qquad \forall A \subseteq \mathbf{N},$$

where I is any subset of $\mathbf{N}$. As $\mathfrak{F}$ is an ultrafilter, $\vartheta_{\mathbf{N}}[\mathfrak{F}] = K\setminus\mathfrak{F}$. At the same time, as $\mathfrak{F}$ is non-principal, $\vartheta_I[\mathfrak{F}] = \mathfrak{F}$ for any finite $I \subseteq \mathbf{N}$. So, for any finite $I \subseteq \mathbf{N}$, $\vartheta_{\mathbf{N}}[E] \cap \vartheta_I[E] = \varnothing$, and

$$\vartheta_{\mathbf{N}}[G] \cap \vartheta_I[G] \subseteq (\vartheta_{\mathbf{N}}[G] \triangle \vartheta_{\mathbf{N}}[E]) \cup (\vartheta_I[G] \triangle \vartheta_I[E])$$

is meager, i.e., $\vartheta_{\mathbf{N}}[G] \cap \vartheta_I[G] = \varnothing$.

On the other hand,

$$\begin{aligned} \bigcup \{\vartheta_I[G] : I \subseteq \mathbf{N}, I \text{ finite}\} & \\ &= \{A : \exists \text{ finite } I \subseteq \mathbf{N}, A \triangle I \in G\} \\ &= K \qquad \text{if} \quad G \neq \varnothing. \end{aligned}$$

So $\vartheta_{\mathbf{N}}[G] = \varnothing$, which is absurd.

This shows that $G = \varnothing$ and E is meager.

The same argument applies if E is a Souslin subset of $K\setminus\mathfrak{F}$.

(b) Essentially the same arguments show that $\mu E=0$. Here the function of the open set G is taken over by a compact set $L\subseteq E$ which is supposed to be of positive measure; we observe that in this case

$$\mu\Big(\bigcup\{\vartheta_I[L]: I\subseteq\mathbf{N}, I \text{ finite}\}\Big) = 1$$

to obtain our contradiction. (This is in effect done in [14].)

(c) It follows at once that $\mathfrak{F}$ is non-measurable and does not have the Baire property. For if $\mathfrak{F}$ were measurable, there would be F_σ subsets of $\mathfrak{F}$ and $K\setminus\mathfrak{F}$ with measures summing to 1; and if $\mathfrak{F}$ had the Baire property, there would be G_δ subsets of $\mathfrak{F}$ and $K\setminus\mathfrak{F}$ with comeager union; but both F_σ and G_δ sets in K are Souslin.

2E. We end this series of lemmas with a final elementary remark.

LEMMA. *Let X and Y be any sets, $\varphi: X\to Y$ a surjection. If $B\subseteq\mathbf{R}^Y$, and $A=\{y\varphi: y\in B\}\subseteq\mathbf{R}^X$, then the map $y\mapsto y\varphi:\mathbf{R}^Y\to\mathbf{R}^X$ induces an affine homeomorphism between the closed convex hulls $\overline{\Gamma(B)}$ and $\overline{\Gamma(A)}$. (Here, as throughout this paper, $\mathbf{R}^X$ and $\mathbf{R}^Y$ are given their pointwise topologies.)*

2F. Theorem. *Let X be any Hausdorff space and $A\subseteq C(X)$ a pointwise bounded set. Then the following are equivalent:*

(i) *for every compact $K\subseteq X$, $A_K=\{x|_K: x\in A\}$ is relatively countably compact in $\hat{\boldsymbol{B}}(K)$ (i.e., every sequence in A_K has a cluster point in $\hat{\boldsymbol{B}}(K)$);*

(ii) *for every compact $K\subseteq X$, A_K is relatively sequentially compact in $\mathbf{R}^K$ (i.e., every sequence in A_K has a pointwise convergent subsequence);*

(iii) *for every Čech-complete $Y\subseteq X$, A_Y is relatively compact in $\boldsymbol{B}^{\S}(Y)$;*

(iv) *A is relatively compact in $\boldsymbol{M}_r(X)$;*

(v) *for every Radon measure μ on X, A is relatively countably compact in $\boldsymbol{M}_\mu(X)$;*

(vi) *if $K\subseteq X$ is compact, $\langle x_n\rangle_{n\in\mathbf{N}}$ is a sequence in A, and $\alpha<\beta$ in $\mathbf{R}$, then there is some $I\subseteq\mathbf{N}$ such that*

$$\{t: t\in K, x_n(t)\leqslant\alpha\ \forall n\in I, x_n(t)\geqslant\beta\ \forall n\in\mathbf{N}\setminus I\} = \varnothing.$$

Proof. For definitions of $\hat{\boldsymbol{B}}$, $\boldsymbol{B}_r^{\S}$, $\boldsymbol{M}_r$ see 1A above.

(a) (iii)⇒(i) and (iv)⇒(v) are evident (for the former, we must remember that $\boldsymbol{B}^{\S}(K)\subseteq\hat{\boldsymbol{B}}(K)$ for any K—see 1C(d) above—and that compact sets are Čech-complete).

(b) Suppose that (vi) fails: that $K \subseteq X$ is a compact set, $\langle x_n \rangle_{n\in\mathbf{N}}$ is a sequence in A, and that $\alpha < \beta$ in $\mathbf{R}$ are such that

$$\{t : t \in K, x_n(t) \leqslant \alpha \,\forall n \in I, x_n(t) \geqslant \beta \,\forall n \in \mathbf{N}\setminus I\} \neq \varnothing$$

for any $I \subseteq \mathbf{N}$. Set

$$K_0 = \{t : t \in K, x_n(t) \notin \,]\alpha, \beta[\, \forall n \in \mathbf{N}\};$$

because every x_n is continuous, K_0 is compact and has the same property as K. Define $\varphi : K_0 \to \mathcal{P}\mathbf{N}$ by writing

$$\varphi t = \{n : x_n(t) \geqslant \beta\}$$

for each $t \in K_0$. Then φ is a continuous surjection.

(α) Let $L \subseteq K_0$ be a set which is minimal subject to the requirements that $\varphi[L] = \mathcal{P}\mathbf{N}$ and L is closed; such a set exists because Zorn's lemma applies to the family of all such subsets of K_0. We show that A_L is not relatively countably compact in $\hat{\boldsymbol{B}}(L)$. For if it were, then $\langle x_n|_L \rangle_{n\in\mathbf{N}}$ would have a cluster point $z \in \hat{\boldsymbol{B}}(L)$, and z would be expressible as $\lim_{n\to\mathcal{F}} x_n|_L$ for some non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$. But, because $L \subseteq K_0$, z can take only values outside the open interval $]\alpha, \beta[$; and we see that, for $t \in L$,

$$z(t) \geqslant \beta \quad \Leftrightarrow \quad \{n : x_n(t) \geqslant \beta\} \in \mathcal{F} \quad \Leftrightarrow \quad \varphi t \in \mathcal{F}.$$

So $L \cap \varphi^{-1}[\mathcal{F}] = \{t : z(t) \geqslant \beta\}$ has the Baire property in L. But Lemma 2C now implies that $\mathcal{F}$ has the Baire property in $\mathcal{P}\mathbf{N}$, which by Lemma 2D is not the case. Thus A_L is not relatively countably compact in $\hat{\boldsymbol{B}}(L)$, and (i) fails.

(β) Let ν be the canonical measure on $\mathcal{P}\mathbf{N}$. Then there is a Radon measure μ on L such that $\mu = \nu\varphi^{-1}$ in the strong sense that a subset E of $\mathcal{P}\mathbf{N}$ is ν-measurable iff $\varphi^{-1}[E]$ is μ-measurable, and the measures then agree (see [9], Chapter I, §1, Theorem 9, p. 35). The argument now follows the lines of (α) above; if z is any cluster point of $\langle x_n \rangle_{n\in\mathbf{N}}$, then $\{t : t \in L, z(t) \geqslant \beta\}$ must be equal to $L \cap \varphi^{-1}[\mathcal{F}]$ for some ultrafilter $\mathcal{F}$ on $\mathbf{N}$; as, by Lemma 2D, $\mathcal{F}$ is not measurable for ν, $L \cap \varphi^{-1}[\mathcal{F}]$ cannot be measurable for μ, and $z \notin \boldsymbol{M}_{\bar{\mu}}(X)$, where $\bar{\mu}$ is the Radon measure on X extending μ. Thus A is not relatively countably compact in $\boldsymbol{M}_{\bar{\mu}}(X)$, and (v) fails.

(c) Thus we see that (i)$\Rightarrow$(vi) and (v)$\Rightarrow$(vi). To prove (vi)$\Rightarrow$(iii), we use Lemma 2B. Let us suppose rather that (iii) fails: that for some Čech-complete $Y \subseteq X$, A_Y is not relatively compact in $\boldsymbol{B}_r^{\S}(Y)$. As A is pointwise bounded, A_Y is certainly relatively compact in $\mathbf{R}^Y$, so this must mean that there is a $z \in \bar{A}_Y \setminus$

$B_r^{\S}(Y)$. By Corollary 1D(a), there is a non-empty (relatively) closed set $F \subseteq Y$ and real numbers $\alpha < \beta$ such that $S = \{t : t \in F, z(t) \leqslant \alpha\}$ and $T = \{t : t \in F, z(t) \geqslant \beta\}$ are both dense in F.

Take γ, δ such that $\alpha < \gamma < \delta < \beta$, and let $\mathcal{Q}$ be the family of pairs (G_x, H_x) as x runs through A, where

$$G_x = \{t : t \in Y, x(t) < \gamma\}, \qquad H_x = \{t : t \in Y, x(t) > \delta\}.$$

Then the G_x and H_x are all (relatively) open in Y. Now $\mathcal{Q}$ is weakly dense over F in the sense of Lemma 2B. For if $E_0, \ldots, E_n$ are (relatively) open sets in Y all meeting F, they must meet both S and T, so there exist $s_0, \ldots, s_n$, $t_0, \ldots, t_n$ such that $s_i \in S \cap E_i$, $t_i \in T \cap E_i$; now $z(s_i) \leqslant \alpha < \gamma < \delta < \beta \leqslant z(t_i)$ for each $i \leqslant n$, and $z \in \bar{A}_Y$, so there must be some $x \in A$ such that $x(s_i) < \gamma < \delta < x(t_i)$ for each $i \leqslant n$; and in this case $G_x \cap F \cap E_i \neq \varnothing$, $H_x \cap F \cap E_i \neq \varnothing$ for each $i \leqslant n$.

Lemma 2B now tells us that there is a compact $K \subseteq Y$ and a sequence in $\mathcal{Q}$ which is completely independent over K; that is, there is a sequence $\langle x_n \rangle_{n \in \mathbf{N}}$ in A such that, for every $I \subseteq \mathbf{N}$,

$$\{t : t \in K, x_n(t) \leqslant \gamma \, \forall n \in I, x_n(t) \geqslant \delta \, \forall n \in \mathbf{N} \setminus I\}$$

$$\supseteq K \cap \bigcap_{n \in I} G_{x_n} \cap \bigcap_{n \in \mathbf{N} \setminus I} H_{x_n} \neq \varnothing.$$

Thus (vi) has also failed. Turning this round, we have (vi)⇒(iii).

(d) Next suppose that (iii) is true. As remarked just above, we know that A is relatively compact in $\mathbf{R}^X$. Let $z \in \bar{A}$ in $\mathbf{R}^X$. Then for any compact $K \subseteq X$, we have $z|_K \in \overline{A_K}$ in $\mathbf{R}^K$. But we are supposing that A_K is relatively compact in $B_r^{\S}(K)$, so $z|_K \in B_r^{\S}(K)$. By Proposition 1F, $z \in M_r(X)$. So $\bar{A} \subseteq M_r(X)$, and A is relatively compact in $M_r(X)$. Thus we see that (iii)⇒(iv).

(e) Putting these together, we see that (i) and (iii)–(vi) are equivalent. Of course (ii)⇒(i), so our remaining task is to prove that (vi) implies (ii).

Let $\langle x_n \rangle_{n \in \mathbf{N}}$ be any sequence in A, and $K \subseteq X$ any compact set. Define $\varphi : K \to \mathbf{R}^{\mathbf{N}}$ by writing $(\varphi t)(n) = x_n(t)$ for $n \in \mathbf{N}$, $t \in K$. As $A \subseteq C(X)$, it follows that φ is continuous, and $L = \varphi[K]$ is compact; moreover, L is metrizable. Let B be the set of coordinate functionals on L, so that $\{y\varphi : y \in B\} = \{x_n|_K : n \in \mathbf{N}\}$.

We find that condition (vi), assumed for A, implies the corresponding property for B. For if $\langle y_n \rangle_{n \in \mathbf{N}}$ is any sequence in B, and $\alpha < \beta$, there is some $I \subseteq \mathbf{N}$ such that

$$\{t : t \in K, y_n\varphi(t) \leqslant \alpha \, \forall n \in I, y_n\varphi(t) \geqslant \beta \, \forall n \in \mathbf{N} \setminus I\} = \varnothing;$$

as $\varphi[K]=L$,

$$\{u: u\in L, y_n(u)\leqslant\alpha\,\forall n\in I, y_n(u)\geqslant\beta\,\forall n\in \mathbf{N}\setminus I\} = \varnothing.$$

So B is relatively compact in $\boldsymbol{B}_r^{\S}(L)$, because we know already that (vi)⇒(iii). But L is Polish, so $\boldsymbol{B}_r^{\S}(L)=\boldsymbol{B}_1(L)$ (Proposition 1E). Thus B is a subset of $\boldsymbol{C}(L)$ which is relatively compact in $\boldsymbol{B}_1(L)$. It follows from the Main Theorem of [12] that B is relatively sequentially compact in $\mathbf{R}^L$. So, as remarked in 2E, $\{x_n|_K : n\in\mathbf{N}\}$ is relatively sequentially compact in $\mathbf{R}^K$, and $\langle x_n|_K\rangle_{n\in\mathbf{N}}$ has a convergent subsequence. As $\langle x_n\rangle_{n\in\mathbf{N}}$ is arbitrary, A_K is relatively sequentially compact in $\mathbf{R}^K$; as K is arbitrary, A satisfies condition (ii), as required.

Remarks. The argument in part (b) above actually shows that the above conditions are equivalent to:

(vii) if $L\subseteq X$ is compact, and $\langle x_n\rangle_{n\in\mathbf{N}}$ is a sequence in A, and μ is a Radon measure on L, then $\langle x_n|_L\rangle_{n\in\mathbf{N}}$ has a cluster point in $\hat{\boldsymbol{B}}(L)\cup \boldsymbol{M}_\mu(L)$.

Possibly there is some more elegant way of expressing the fact that if A does not satisfy these conditions, then it contains a sequence such that every cluster point is hopelessly non-measurable.

For further equivalent conditions to add to the list, see 4E, 4G and 5H below.

3. Compact Sets in $B_1(X)$.

Returning now to the problems attacked in [12], we show that the first Baire class of functions on a Polish space is actually "angelic" in the sense of [10].

3A. Definition. A regular Hausdorff space is *angelic* if (i) every relatively countably compact set is relatively compact, (ii) the closure of a relatively compact set is precisely the set of limits of its sequences.

Useful facts about angelic spaces include: (a) any subspace of an angelic space is angelic; (b) if $\mathfrak{T}$ is an angelic topology on a set X, and $\mathfrak{S}$ is any finer regular topology on X, then $\mathfrak{S}$ is angelic; (c) in an angelic space, compact sets are sequentially compact (so that relatively countably compact sets are relatively sequentially compact); (d) if X is a topological space such that there is a sequence $\langle K_n\rangle_{n\in\mathbf{N}}$ of countably compact subsets of X such that $\bigcup_{n\in\mathbf{N}}K_n$ is dense in X, then $C(X)$ is angelic under $\mathfrak{T}_p$ (this applies, in particular, if X is separable). For proofs of these results, with others, see [9].

In [12], H. P. Rosenthal has shown that $B_1(X)$, where X is a Polish space, satisfies the condition (i) above; next, that if $A \subseteq B_1(X)$ is relatively compact, then A is relatively sequentially compact [(c) above]; and finally that if $A \subseteq B_1(X)$ is relatively compact and $x \in \bar{A}$, then there is a countable $B \subseteq A$ such that $x \in \bar{B}$. We shall rely on these results in completing the proof that $B_1(X)$ is angelic.

3B. We begin with a simple topological lemma.

Lemma. *Let X be a regular Hausdorff space which is sequentially compact and such that if $A \subseteq X$ and $x \in \bar{A}$, there is a countable $A_0 \subseteq A$ such that $x \in \bar{A}_0$. Let $\langle x_n \rangle_{n \in \mathbf{N}}$ be a sequence in X and $\langle I_n \rangle_{n \in \mathbf{N}}$ a decreasing sequence of infinite subsets of N, such that the subsequences $\langle x_i \rangle_{i \in I_n}$ have a common cluster point x. Then there is an infinite set $I \subseteq \mathbf{N}$ such that $I \setminus I_n$ is finite for every $n \in \mathbf{N}$ and x is a cluster point of $\langle x_i \rangle_{i \in I}$.*

Proof. (a) Set

$$F = \left\{ \lim_{i \in I} x_i : I \text{ is an infinite set such that} \lim_{i \in I} x_i \text{ exists and } I \setminus I_n \text{ is finite } \forall n \in \mathbf{N} \right\}.$$

Then $x \in \bar{F}$. For if U is any neighborhood of x, then

$$J = \{ i : i \in \mathbf{N}, x_i \in U \}$$

meets every I_n in an infinite set. As $\langle I_n \rangle_{n \in \mathbf{N}} \downarrow$, there is an infinite $K \subseteq J$ such that $K \setminus I_n$ is finite for every $n \in \mathbf{N}$. Now X is sequentially compact, so there is an infinite $I \subseteq K$ such that $z = \lim_{i \in I} x_i$ exists. In this case $z \in F \cap \bar{U}$. As X is regular and U is arbitrary, $x \in \bar{F}$.

(b) By the other half of our hypothesis on X, there is a sequence $\langle z_m \rangle_{m \in \mathbf{N}}$ in F such that $x \in \overline{\{ z_m : m \in \mathbf{N} \}}$. Express each z_m as $\lim_{i \in J_m} x_i$, where J_m is an infinite set such that $J_m \setminus I_n$ is finite for every $n \in \mathbf{N}$. Set $I = \bigcup_{n \in \mathbf{N}} (I_n \cap J_n)$. Then it is easy to see (because the I_n are decreasing) that $I \setminus I_n$ and $J_m \setminus I$ are finite for every $m, n \in \mathbf{N}$. It follows that every z_m is a cluster point of $\langle x_i \rangle_{i \in I}$; as the set of cluster points of a sequence is always closed, x is also a cluster point of $\langle x_i \rangle_{i \in I}$. Thus I is the required set.

3C. We now embark on a series of lemmas culminating in Theorem 3C.

Lemma. *Let X be a Polish space, and $\langle x_n \rangle_{n \in \mathbf{N}}$ a sequence in $C(X)$ such that* (i) *$\{ x_n : n \in \mathbf{N} \}$ is relatively compact in $B_1(X)$,* (ii) *0 is a cluster point of*

$\langle x_n \rangle_{n\in\mathbf{N}}$ *(the topology* $\mathfrak{T}_p$*, as always, being understood throughout). Let* $W \subseteq X$ *be a non-empty closed set, and* $\varepsilon > 0$*. Then there is a non-empty relatively open set* $U \subseteq W$ *and an infinite set* $J \subseteq \mathbf{N}$ *such that* 0 *is a cluster point of* $\langle x_i \rangle_{i\in J}$ *and*

$$\limsup_{i\in J} |x_i(t)| \leqslant 2\varepsilon \qquad \forall t \in U.$$

Proof. For any infinite set $I \subseteq \mathbf{N}$, let $A(I)$ be the set of cluster points of $\langle x_i \rangle_{i\in I}$, a non-empty set in $\boldsymbol{B}_1(X)$. For finite I, set $A(I) = \varnothing$.

Suppose, if possible, that the result is false. Let $\mathcal{Q}$ be the set $\{(G_i, H_i) : i \in \mathbf{N}\}$, where

$$G_i = \{t : t \in X, |x_i(t)| < \varepsilon\}, \qquad H_i = \{t : t \in X, |x_i(t)| > 2\varepsilon\}.$$

Then $\mathcal{Q}$ is weakly dense over W in the sense of Lemma 2B. For let $E_0, \ldots, E_n$ be open sets in X meeting W. Take any points $s_0, \ldots, s_n$ in $E_0 \cap W, \ldots, E_n \cap W$, and set $I = \{i : |x_i(s_r)| < \varepsilon \ \forall r \leqslant n\}$. Then surely $0 \in A(I)$. Next, setting

$$J_r = \{i : i \in I, |x_i(t)| \leqslant 2\varepsilon \ \forall t \in E_r \cap W\},$$

our hypothesis is that $0 \notin A(J_r)$ for any $r \leqslant n$; but it follows that $I \neq \bigcup_{r\leqslant n} J_r$, for certainly $A(\bigcup_{r\leqslant n} J_r) = \bigcup_{r\leqslant n} A(J_r)$. If i is any point of $I \setminus \bigcup_{r\leqslant n} J_r$, then we see that $G_i \cap E_r \cap W \neq \varnothing$ (as $i \in I$), $H_i \cap E_r \cap W \neq \varnothing$ (as $i \notin J_r$) for every $r \leqslant n$.

Consequently, there is a compact $K \subseteq X$ and a sequence in $\mathcal{Q}$ which is completely independent over K in the sense of Lemma 2B; i.e., there is a sequence $\langle y_n \rangle_{n\in\mathbf{N}}$ in $\{x_i : i \in \mathbf{N}\}$ such that, for every $I \subseteq \mathbf{N}$,

$$\{t : t \in K, |y_n(t)| < \varepsilon \ \forall n \in I, |y_n(t)| > 2\varepsilon \ \forall n \in \mathbf{N} \setminus I\}$$

is not empty. It follows that $\langle |y_n| \rangle_{n\in\mathbf{N}}$ can have no convergent subsequence, and therefore that $\langle y_n \rangle_{n\in\mathbf{N}}$ has no convergent subsequence. But $\langle y_n \rangle_{n\in\mathbf{N}}$ is a sequence in $\{x_i : i \in \mathbf{N}\}$ which is relatively compact, and therefore relatively sequentially compact, in $\boldsymbol{B}_1(X)$. This contradiction proves the result.

3D. Lemma. *Under the conditions of Lemma* 3C, *there is an infinite set* $I \subseteq \mathbf{N}$ *such that* $\limsup_{i\in I} |x_i(t)| \leqslant \varepsilon$ *for every* $t \in X$, *and* 0 *is a cluster point of* $\langle x_i \rangle_{i\in I}$.

Proof. (a) For each infinite set $I \subseteq \mathbf{N}$, set

$$U(I) = \operatorname{int}\left\{t : \limsup_{i\in I} |x_i(t)| \leqslant \varepsilon\right\},$$

and as before let $A(I)$ be the set of cluster points of $\langle x_i\rangle_{i\in I}$. Observe that if $I\setminus J$ is finite, then $U(I)\supseteq U(J)$. Enumerate a base for the topology of X as $\langle V_k\rangle_{k\in\mathbf{N}}$ (passing over the trivial case in which X is finite), and choose I_k as follows. $I_0=\mathbf{N}$. Given I_k such that $0\in A(I_k)$, then if there is an infinite $I\subseteq I_k$ such that $0\in A(I)$ and $V_k\subseteq U(I)$, take such a set for I_{k+1}; otherwise, set $I_{k+1}=I_k$.

(b) Thus $\langle I_k\rangle_{k\in\mathbf{N}}$ is a decreasing sequence of sets such that $0\in A(I_k)$ for every $k\in\mathbf{N}$. Applying Lemma 3B to the set $\overline{\{x_i: i\in\mathbf{N}\}}$, which by Rosenthal's theorems satisfies the conditions, we see that there is an infinite set $I\subseteq\mathbf{N}$ such that $0\in A(I)$ and $I\setminus I_k$ is finite for every $k\in\mathbf{N}$.

Let J be any infinite subset of I such that $0\in A(J)$. Then of course $U(J)\supseteq U(I)$. Suppose, if possible, that $U(J)\neq U(I)$. Then there is a $k\in\mathbf{N}$ such that $V_k\subseteq U(J)$ but $V_k\not\subseteq U(I)$. Now $J\setminus I_k$ is finite, so $J\cap I_k$ is an infinite subset of I_k such that $0\in A(J\cap I_k)$ and $V_k\subseteq U(J\cap I_k)$. Accordingly, at this stage in the inductive definition, we must have chosen I_{k+1} such that $V_k\subseteq U(I_{k+1})$. But in this case $I\setminus I_{k+1}$ is finite, so that $V_k\subseteq U(I)$, contrary to our hypothesis.

This shows that $U(J)=U(I)$ for every subset J of I such that $0\in A(J)$.

(c) Now suppose, if possible, that $U(I)\neq X$. Let W be the closed set $X\setminus U(I)$. By Lemma 3C, applied to the sequence $\langle x_i\rangle_{i\in I}$, there is an infinite $J\subseteq I$ such that $0\in A(J)$ and $\limsup_{i\in J}|x_i(t)|\leqslant\varepsilon$ for every $t\in U$, where $U\subseteq W$ is relatively open; so that

$$\limsup_{i\in J}|x_i(t)|\leqslant\varepsilon\qquad\forall t\in U\cup U(I),$$

and $U(J)\subseteq\operatorname{int}[U\cup U(I)]\neq U(I)$; which is impossible, by (b).

(d) Thus $U(I)=X$ and $\limsup_{i\in I}|x_i(t)|\leqslant\varepsilon$ for every $t\in X$, as required.

3E. Corollary. *Under the conditions of Lemma* 3C, *there is a subsequence of* $\langle x_n\rangle_{n\in\mathbf{N}}$ *converging to* 0.

Proof. Applying Lemma 3D repeatedly, we can choose successively smaller infinite sets $I_k\subseteq\mathbf{N}$ such that $I_0=\mathbf{N}$, 0 is a cluster point of $\langle x_i\rangle_{i\in I_k}$, and

$$\limsup_{i\in I_k}|x_i(t)|\leqslant 2^{-k}\qquad\forall t\in X,\ k\geqslant 1.$$

If we now take any infinite set $I\subseteq\mathbf{N}$ such that $I\setminus I_k$ is finite for every k, we see that $0=\lim_{n\in I}x_n$.

3F. Theorem. *If X is a Polish space, then $\boldsymbol{B}_1(X)$ is angelic under $\mathfrak{T}_p$.*

Proof. We have only to prove that if $A\subseteq\boldsymbol{B}_1(X)$ is relatively compact, and $x\in\overline{A}$, then there is a sequence in A converging to x. We know that x is in the closure of some countable subset of A [12, Main Theorem], i.e., that there is a sequence $\langle x_n \rangle_{n\in\mathbf{N}}$ in A such that x is a cluster point of $\langle x_n \rangle_{n\in\mathbf{N}}$.

Define $\varphi : X \to \mathbf{R}^{\mathbf{N}}$ by writing

$$(\varphi t)(0) = x(t),$$
$$(\varphi t)(n+1) = x_n(t) \qquad \forall n \in \mathbf{N},\ t \in X.$$

Then φ is Borel measurable, so $Y = \varphi[X]$ is Souslin [9, §38, Part III, No. 5]. If we set $y(u) = u(0)$, $y_n(u) = u(n+1)$ for $u \in Y$, we see that y is a cluster point of $\langle y_n \rangle_{n\in\mathbf{N}}$ in $\mathbf{R}^Y$, and further that every subsequence of $\langle y_n \rangle_{n\in\mathbf{N}}$ has a convergent sub-subsequence [because A is relatively sequentially compact in $\boldsymbol{B}_1(X)$, so that every subsequence of $\langle x_n \rangle_{n\in\mathbf{N}}$ has a convergent sub-subsequence].

Now let Z be a Polish space and $\psi : Z \to Y$ a continuous surjection. Set $z = y\psi$, $z_n = y_n\psi$ for $n \in \mathbf{N}$. Then z is a cluster point of $\langle z_n \rangle_{n\in\mathbf{N}}$ in $\mathbf{R}^{Z}$, and $\langle z_n \rangle_{n\in\mathbf{N}}$ is a sequence in $\boldsymbol{C}(Z)$ such that every subsequence has a convergent sub-subsequence. By the results of [12], $\{z_n : n \in \mathbf{N}\}$ is relatively compact in $\boldsymbol{B}_1(Z)$. Now z also belongs to $\boldsymbol{C}(Z)$, so we may apply Corollary 3E to $\langle z_n - z \rangle_{n\in\mathbf{N}}$ to see that there is a subsequence $\langle z_n \rangle_{n\in I}$ converging to z. At once we have $\lim_{n\in I} y_n = y$ and $\lim_{n\in I} x_n = x$, as required.

Remark. This gives a new proof of part (b) of the Main Theorem of [12], because any sequentially continuous function on a relatively compact set in $\boldsymbol{B}_1(X)$ will be continuous.

3G. Corollary. *Let X be a Souslin space. Then $\boldsymbol{B}_1(X)$ is angelic.*

Proof. Let Z be a Polish space and $\varphi : Z \to X$ a continuous surjection. Then the map $x \mapsto x\varphi : \boldsymbol{B}_1(X) \to \boldsymbol{B}_1(Z)$ is a homeomorphism between $\boldsymbol{B}_1(X)$ and a subset of $\boldsymbol{B}_1(Z)$. As $\boldsymbol{B}_1(Z)$ is angelic, so is $\boldsymbol{B}_1(X)$, by 3A(a).

4. Compact Sets in $\boldsymbol{Ba}(X)$.

In this section we describe some of the consequences of Theorems 2F and 3F. If X is a Polish space, then the same technique we have already used in Theorem 3F shows that any separable, relatively countably compact set in $\boldsymbol{B}(X)$ can be represented as a subset of $\boldsymbol{B}_1(Z)$ for some other Polish space Z [see part (c) of the proof of 4D]. Consequently we can apply our results about $\boldsymbol{B}_1$ to separable sets of Borel-measurable functions.

The same methods work for $\boldsymbol{Ba}(X)$ [definition: 1A(h)] if X is compact and Hausdorff. So once again we seek a common generalization. In this case it seems that the appropriate concept is that of "K-analytic" space, as described in [2].

4A. *K*-Analytic Spaces. Following [2], we shall say that a $K'_{\sigma\delta}$ set, in any topological space, is one expressible as $\bigcap_{n\in\mathbf{N}}\bigcup_{m\in\mathbf{N}}K_{nm}$, where each K_{nm} is compact and closed. Now a *K-analytic* space is a Hausdorff space which is a continuous image of a $K'_{\sigma\delta}$ subset of some other topological space. We shall call a subset of a topological space K-analytic if it is K-analytic in its subspace topology.

We shall require the following facts:

(a) if X is a K-analytic space, a subset Y of X is K-analytic iff it can be obtained by the Souslin operation from closed subsets of X;
(b) the product of finitely many K-analytic spaces is K-analytic;
(c) a metrizable space is K-analytic iff it is Souslin;
(d) if X is any Hausdorff topological space, any K-analytic subset of X is universally measurable.

For proofs of these, see [2], [13, Part I, Chapter 2, §3], and [3, §8].

4B. *K*-Analytic Functions. In order to cover simultaneously (i) the family of Borel-measurable functions on a Souslin space and (ii) the family of Baire measurable functions on a compact Hausdorff space, we use the following idea. We begin by observing that if X is a K-analytic space, Y is a separable metric space, and $f:X\to Y$ is a function, then the following are equivalent:

(α) the graph of f in $X\times Y$ is a K-analytic set;
(β) $f^{-1}[G]$ is a K-analytic set in X for every open set $G\subseteq Y$;
(γ) f is measurable for the σ-algebra

$$\Sigma=\{E:E\subseteq X,\ E\text{ and }X\setminus E\text{ are both }K\text{-analytic}\}$$

of subsets of X.

(The result remains true if Y is not taken to be separable, but the proof becomes harder.) Such functions we shall call *K-analytic functions*, and the space of K-analytic real-valued functions on a K-analytic space X will be denoted $\boldsymbol{Ka}(X)$. Note that:

(a) $\boldsymbol{Ka}(X)$ is a σ-sublattice of $\mathbf{R}^X$ (because Σ is a σ-subalgebra of $\mathcal{P}X$);
(b) $\boldsymbol{C}(X)\subseteq\boldsymbol{Ba}(X)\subseteq\boldsymbol{Ka}(X)\subseteq\boldsymbol{B}(X)$ (because disjoint K-analytic sets can be separated by Borel sets; see [2], Theorem 5.8);
(c) if X is completely regular, then $\boldsymbol{Ka}(X)=\boldsymbol{Ba}(X)$ (because, in this case, disjoint K-analytic sets can be separated by sets in the σ-algebra generated by the zero sets);

(d) if X is Souslin, $\boldsymbol{Ka}(X)=\boldsymbol{B}(X)$;
(e) if $Y\subseteq X$ is K-analytic, and $x\in\boldsymbol{Ka}(X)$, then $x|_Y\in\boldsymbol{Ka}(Y)$.

4C. Lemma. *Let X be a K-analytic space, Y a separable metric space, and $\varphi:X\to Y$ a K-analytic surjection. If ν is a finite Radon measure on Y, there is a Radon measure μ on X such that $\nu=\mu\varphi^{-1}$ in the strong sense that a set $E\subseteq Y$ is a ν-measurable iff $\varphi^{-1}[E]$ is μ-measurable, and in this case $\nu E=\mu\varphi^{-1}[E]$.*

Proof. Let Γ be the graph of φ in $X\times Y$. Then Γ is K-analytic. By [13, Part I, Chapter 2, Theorem 12] there is a Radon measure λ on Γ such that $\nu=\lambda\pi_2^{-1}$, where $\pi_2:\Gamma\to Y$ is the canonical map. Now $\mu=\lambda\pi_1^{-1}$ is a Radon measure on X.

If $E\subseteq Y$ is a Borel set, then

$$\mu\varphi^{-1}[E]=\lambda\pi_1^{-1}\big[\varphi^{-1}[E]\big]=\lambda(\varphi\pi_1)^{-1}[E]=\lambda\pi_2^{-1}[E]=\nu E.$$

Because Y is separable and metrizable, φ must be Lusin μ-measurable [13, Part I, Chapter 1, Theorem 5]; so the fact that $\nu=\mu\varphi^{-1}$ in the strong sense given is a corollary of *ibid.*, Theorem 9.

4D. Theorem. *Let X be a K-analytic space, and $A\subseteq\boldsymbol{Ka}(X)$ a countable pointwise bounded set. Then the following are equivalent:*

(i) *A is relatively sequentially compact in $\mathbf{R}^X$;*
(ii) *for every finite Radon measure μ on X, A is relatively countably compact in $\boldsymbol{M}_\mu(X)$;*
(iii) *for every compact $K\subseteq X$, $A_K=\{x|_K:x\in A\}$ is relatively countably compact in $\hat{\boldsymbol{B}}(K)$;*
(iv) *A is relatively compact in $\boldsymbol{Ka}(X)$;*
(v) *A is relatively countably compact in $\boldsymbol{B}(X)$;*
(vi) *the closure of A in $\mathbf{R}^X$ is angelic;*
(vii) *the closure of A in $\mathbf{R}^X$ has cardinal less than $2^{\mathfrak{c}}$;*
(viii) *if $\alpha<\beta$ and $\langle x_n\rangle_{n\in\mathbf{N}}$ is a sequence in A, then there is an $I\subseteq\mathbf{N}$ such that*

$$\{t:t\in X,\, x_n(t)\leqslant\alpha\ \forall n\in I,\, x_n(t)\geqslant\beta\ \forall n\in\mathbf{N}\setminus I\}=\varnothing.$$

Proof. We shall prove (viii)⇒(iii)⇒(ii)⇒(vi)⇒(vii)⇒(viii), (vi)⇒(i)⇒(v)⇒(iii), and (vi)⇒(iv)⇒(v).

(a) (viii)⇒(iii): Suppose that (viii) holds, and that $K \subseteq X$ is compact. As A_K is a countable set in $\boldsymbol{Ka}(K) \subseteq \boldsymbol{B}(K) \subseteq \hat{\boldsymbol{B}}(K)$, there is a dense, relatively G_δ set $Y \subseteq K$ such that $A_Y \subseteq \boldsymbol{C}(Y)$. Now Y, being a G_δ set in a compact Hausdorff space, is Čech-complete. So Theorem 2F (vi)⇒(iii) tells us that A_Y is relatively compact in $\boldsymbol{B}_r^{\S}(Y) \subseteq \hat{\boldsymbol{B}}(Y)$. But, as Y is comeager in K,

$$\hat{\boldsymbol{B}}(K) = \{x : x \in \mathbf{R}^K, x|_Y \in \hat{\boldsymbol{B}}(Y)\},$$

and A_K is relatively compact in $\hat{\mathbf{B}}(K)$.

(b) (iii)⇒(ii): Suppose that (iii) holds, and that μ is a finite Radon measure on X. As A is countable and $A \subseteq \boldsymbol{M}_\mu(X)$, there is an increasing sequence $\langle K_n \rangle_{n \in \mathbf{N}}$ of compact subsets of X such that $A_{K_n} \subseteq \boldsymbol{C}(K_n)$ for each $n \in \mathbf{N}$, and $\mu(X \setminus \bigcup_{n \in \mathbf{N}} K_n) = 0$. Now Theorem 2F (i)⇒(ii) shows that each A_{K_n} is relatively sequentially compact in $\mathbf{R}^{K_n}$. So a diagonal argument shows that any sequence $\langle x_n \rangle_{n \in \mathbf{N}}$ in A will have a subsequence $\langle x_n \rangle_{n \in I}$ which is convergent at each point of $\bigcup_{n \in \mathbf{N}} K_n$. Now any cluster point of $\langle x_n \rangle_{n \in I}$ in $\mathbf{R}^X$ will be a cluster point of $\langle x_n \rangle_{n \in \mathbf{N}}$ in $\boldsymbol{M}_\mu(X)$.

(c) (ii)⇒(vi): Suppose that (ii) holds. Define $\varphi : X \to \mathbf{R}^A$ by writing $(\varphi t)(x) = x(t)$ for $x \in A$, $t \in X$. Then the inverse image of any basic open set in $\mathbf{R}^A$ is K-analytic; as A is countable, we see that $\mathbf{R}^A$ is Polish, and φ is a K-analytic function. Set $Y = \varphi[X]$, so that Y is a metrizable K-analytic space, and therefore Souslin.

For $x \in A$, let $\delta_x : Y \to \mathbf{R}$ be the corresponding coordinate functional on Y. Set $B = \{\delta_x : x \in A\} \subseteq \boldsymbol{C}(Y)$. Then $A = \{y\varphi : y \in B\}$.

Now B satisfies the criterion (v) of Theorem 2F. For let ν be a Radon measure on Y. Then ν is σ-finite, because Y is Lindelöf, so there is a finite Radon measure λ such that $\boldsymbol{M}_\nu(Y) = \boldsymbol{M}_\lambda(Y)$. By Lemma 4C, there is a finite Radon measure μ on X such that $\lambda = \mu\varphi^{-1}$ in the strong sense. Let $\langle y_n \rangle_{n \in \mathbf{N}}$ be any sequence in B. Then $\langle y_n \varphi \rangle_{n \in \mathbf{N}}$ is a sequence in A which has a cluster point in $\boldsymbol{M}_\mu(X)$; say $x = \lim_{n \to \mathfrak{F}} y_n \varphi \in \boldsymbol{M}_\mu(X)$, where $\mathfrak{F}$ is a non-principal ultrafilter on $\mathbf{N}$. If we set $y = \lim_{n \to \mathfrak{F}} y_n$, we see that $x = y\varphi$, so that $y \in \boldsymbol{M}_\lambda(Y) = \boldsymbol{M}_\nu(Y)$ is the required cluster point of $\langle y_n \rangle_{n \in \mathbf{N}}$.

Take a Polish space Z and a continuous surjection $\psi : Z \to Y$. Let $C = \{y\psi : y \in B\} \subseteq \boldsymbol{C}(Z)$. As B satisfies condition (vi) of Theorem 2F, so does C. So C is relatively compact in $\boldsymbol{B}_r^{\S}(Z) = \boldsymbol{B}_1(Z)$ by Theorem 2F (vi)⇒(iii). Accordingly the closure $\bar{C}$ of C in $\mathbf{R}^Z$ is angelic (Theorem 3F). But (see Lemma 2E) $\bar{C}$ is homeomorphic to $\bar{B}$ and therefore to $\bar{A}$, the closures being taken in $\mathbf{R}^Y$ and $\mathbf{R}^X$ respectively. So $\bar{A}$ is angelic.

(d) (vi)⇒(vii) is elementary; for if $\bar{A}$ is angelic, then every point of $\bar{A}$ is the limit of a sequence in A; as A is countable, $\mathrm{card}(\bar{A}) \leqslant \mathfrak{c}$.

(e) (vii)⇒(viii): Suppose rather that (viii) fails; that there is a sequence $\langle x_n \rangle_{n\in\mathbf{N}}$ in A and $\alpha<\beta$ such that

$$D(I) = \{t : x_n(t) \leqslant \alpha\ \forall n\in I,\ x_n(t) \geqslant \beta\ \forall n\in\mathbf{N}\setminus I\} \neq \varnothing$$

for every $I\subseteq\mathbf{N}$. In this case, if $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters on $\mathbf{N}$, there is an $I\subseteq\mathbf{N}$ such that $I\in\mathcal{F}$, $\mathbf{N}\setminus I\in\mathcal{G}$; so that $\lim_{n\to\mathcal{F}}x_n$ and $\lim_{n\to\mathcal{G}}x_n$ differ on $D(I)$. As there are $2^{\mathfrak{c}}$ distinct ultrafilters on $\mathbf{N}$, $\langle x_n \rangle_{n\in\mathbf{N}}$ has $2^{\mathfrak{c}}$ distinct cluster points, and (vii) fails.

(f) The rest is elementary. (vi)⇒(i) because the closure of A in $\mathbf{R}^X$ is certainly compact; so if it is angelic, it will be sequentially compact. (i)⇒(v) because the limit of any sequence in A will belong to $\boldsymbol{Ka}(X)$ and therefore to $\boldsymbol{B}(X)$. (v)⇒(iii) because $\boldsymbol{B}(X)\subseteq\{x : x\in\mathbf{R}^X,\ x|_K\in\hat{\boldsymbol{B}}(K)\ \forall \text{ compact } K\subseteq X\}$. (vi)⇒(iv) because, if the closure of A in $\mathbf{R}^X$ is angelic, then every point of $\bar{A}$ is the limit of a sequence in A, and so belongs to $\boldsymbol{Ka}(X)$. And (iv)⇒(v) because $\boldsymbol{Ka}(X)\subseteq\boldsymbol{B}(X)$.

Note. See also 4H, 5I below.

4E. Corollary. *Let X be any Hausdorff topological space, $A\subseteq\mathbf{C}(X)$ a pointwise bounded subset. Then we may add to the conditions listed in Theorem 2F the following*:

(ii′) *for every K-analytic $Y\subseteq X$, A_Y is relatively sequentially compact in $\mathbf{R}^Y$.*

Proof. Of course, (ii′)⇒(ii). For (i)⇒(ii′), we know that $A_Y\subseteq\mathbf{C}(Y)\subseteq\boldsymbol{Ka}(Y)$. So Theorem 4D (iii)⇒(i) tells us that any countable subset of A_Y is relatively sequentially compact in $\mathbf{R}^Y$, which is exactly the same as saying that A_Y is relatively sequentially compact in $\mathbf{R}^Y$.

4F. Corollary. *Let X be any K-analytic space, $A\subseteq\boldsymbol{Ka}(X)$ a pointwise bounded set. Then the following are equivalent*:

(i) *A is relatively sequentially compact in $\mathbf{R}^X$;*
(ii) *for every finite Radon measure μ on X, A is relatively countably compact in $\boldsymbol{M}_\mu(X)$;*
(iii) *for every compact $K\subseteq X$, A_K is relatively countably compact in $\hat{\boldsymbol{B}}(K)$;*
(iv) *A is relatively countably compact in $\boldsymbol{Ka}(X)$;*
(v) *A is relatively countably compact in $\boldsymbol{B}(X)$;*
(vi) *if $\alpha<\beta$ and $\langle x_n \rangle_{n\in\mathbf{N}}$ is a sequence in A, then there is an $I\subseteq\mathbf{N}$ such*

that

$$\{t : x_n(t) \leqslant \alpha \ \forall n \in I, x_n(t) \geqslant \beta \ \forall n \in \mathbf{N} \setminus I\} = \emptyset.$$

Proof. For these are all equivalent to the requirement that the conditions of Theorem 4D should be satisfied by every countable subset of A.

4G. Corollary. *Let X be a Souslin space, and $A \subseteq \boldsymbol{C}(X)$ a pointwise bounded set. Then the following are equivalent:*

(i) *A satisfies the conditions of Theorem 2F;*
(ii) *A is relatively compact in $\boldsymbol{B}_1(X)$;*
(iii) *A is relatively countably compact in $\boldsymbol{B}(X)$;*
(iv) *A is relatively sequentially compact in $\mathbf{R}^X$;*
(v) *every countable subset of A satisfies the conditions of Theorem 4D.*

Proof. (i)⇔(v) is obvious, because 2F(i) and 4D(iii) correspond. Because X is K-analytic, Corollary 4E gives us (i)⇒(iv); as $\boldsymbol{B}_1(X)$ is angelic (Corollary 3G) and contains all limits of sequences in A, (iv)⇒(ii); (ii)⇒(iii) because $\boldsymbol{B}_1(X) \subseteq \boldsymbol{B}(X)$; and (iii)⇒(v) by 4D(v).

4H. Remarks. The original problem which stimulated this work was the following: if $A \subseteq \boldsymbol{B}([0,1])$ is relatively countably compact, is it relatively sequentially compact? Since $\boldsymbol{B}([0,1]) = \boldsymbol{Ka}([0,1])$, the answer is yes. When X is a Polish space, Theorem 4D becomes a theorem about countable sets in $\boldsymbol{B}(X)$, and 4D(iv) and 4D(v) collapse together; but for compact Hausdorff spaces we have $\boldsymbol{Ka}(X) = \boldsymbol{Ba}(X)$, which may be smaller than $\boldsymbol{B}(X)$ (see 4J below).

It is clear from parts (a) and (b) of the proof of 4D that the conditions of 4D are equivalent to

(ix) whenever $Y \subseteq X$ is such that $A_Y \subseteq \boldsymbol{C}(Y)$, then A_Y satisfies the conditions of Theorem 2F.

However, we note that in 2F and 4G, and also in 3F, there are no countability conditions of the sort that appear in 4D and 4F. It is tempting to suppose that the "uncountable" results might extend outside the first Baire class, but the examples that follow show that this is not true in any simple sense. Theorem 4D itself collapses utterly for uncountable sets A. However, some natural questions remain outstanding: see 4K below.

4I. Example. Let X be the unit interval $[0,1]$. (a) Let A be the set $\{\chi E : E \subseteq X, E \text{ finite}\} \subseteq \boldsymbol{B}_1(X)$. Then A satisfies the conditions of Corollary 4F;

in fact A is relatively sequentially compact in $\boldsymbol{B}_2(X)$, the second Baire class. But of course A is not relatively compact in any of the spaces of functions which we have considered except $\mathbf{R}^X$ itself. Thus $\boldsymbol{B}_2(X)$, unlike $\boldsymbol{B}_1(X)$ and $\boldsymbol{B}_0(X)=\boldsymbol{C}(X)$, is not angelic. (b) If the continuum hypothesis is true, we can enumerate X as $\langle \alpha_\xi \rangle_{\xi<\Omega}$, where Ω is the first uncountable ordinal; now set $x_\xi = \chi\{\alpha_\eta : \eta < \xi\}$ for $\xi \leqslant \Omega$. The map $\xi \mapsto x_\xi : [0,\Omega] \to \mathbf{R}^X$ is continuous, so $A = \{x_\xi : \xi \leqslant \Omega\}$ is compact and homeomorphic to $[0,\Omega]$. In this case A satisfies all the conditions of Theorem 4D except the countability requirement and (vi), and $A \subseteq \boldsymbol{B}_2(X)$. (c) If the continuum hypothesis fails, we can still construct subsets of $\boldsymbol{B}(X)$ homeomorphic to $[0,\Omega]$, though with more difficulty; but we do not know whether this can always be done in a bounded Baire class. (d) Whether the continuum hypothesis is true or not, there is a subset of $[0,1]$ of cardinal $\aleph_1$ which is not Borel-measurable; now the method of (b) gives us a set $A \subseteq \boldsymbol{B}_2(X)$, homeomorphic to $[0,\Omega[$ and therefore sequentially compact, which is not relatively compact in $\boldsymbol{B}(X)$.

4J. Example. Let $X=[0,\Omega]$, where Ω is the first uncountable ordinal, and give X its compact Hausdorff order topology. Because every non-empty set in X has an isolated point, $\boldsymbol{B}_r^{\S}(X)=\mathbf{R}^X$. On the other hand,

$$\boldsymbol{Ka}(X) = \boldsymbol{Ba}(X) = \left\{ x : \lim_{\xi\to\Omega} x(\xi) \text{ exists} = x(\Omega) \right\} = \boldsymbol{B}_1(X);$$

while $\boldsymbol{B}(X)$ is the set of functions $x \in \mathbf{R}^X$ such that there is an uncountable closed set in X on which x is constant. (a) Let $A=\{x : x\in \boldsymbol{C}(X),\ \|x\|_\infty \leqslant 1\}$. Then A satisfies the conditions of Theorem 2F; in particular, A is relatively sequentially compact in $\boldsymbol{B}_1(X)$; but of course A is not relatively compact in $\boldsymbol{B}(X)$. Thus there is no angelic subspace of $\mathbf{R}^X$ including $\boldsymbol{B}_1(X)$. Of the conditions of Theorem 4D, A satisfies (i), (ii), (iii), (v) and (viii), but fails (iv) and (vi) and the countability condition, and may or may not satisfy (vii) [as $\operatorname{card}(\bar{A})=2^\Omega$]. Observe that it also fails (iii) of 4G. (b) Let $A_0 \subseteq A$ comprise the monotonic functions. Then A_0 is relatively compact in $\boldsymbol{B}(X)$, though not in $\boldsymbol{Ka}(X)$, and $\operatorname{card}(\bar{A}_0)=\mathfrak{c}$.

4K. Problems. (a) Suppose that X is Souslin. In Corollary 4G, we have seen that if $A \subseteq \boldsymbol{C}(X)$ is relatively compact in $\boldsymbol{B}(X)$, then it is relatively compact in $\boldsymbol{B}_1(X)$. Are there any other results on these lines? It is easy to see [taking X to be, for instance, the Stone-Čech compactification of an uncountable discrete space Y, and A to be the unit ball of $c_0(Y)$] that this is not literally true for compact Hausdorff spaces X. But suppose we have X K-analytic and $A \subseteq \boldsymbol{C}(X)$ relatively compact in $\boldsymbol{Ka}(X)$; what can we say about $\bar{A}$?

In another direction, suppose that X is Polish and that $A \subseteq \boldsymbol{B}_1(X)$ is relatively compact in $\boldsymbol{B}(X)$; does it follow that $\bar{A} \subseteq \boldsymbol{B}_2(X)$?

(b) Can any part of Theorem 4D be applied to $\boldsymbol{B}(X)$, where X is a compact Hausdorff space? In particular, if X is compact and Hausdorff, and $A \subseteq \boldsymbol{B}(X)$ is a countable, relatively countably compact set, is A relatively compact in $\boldsymbol{B}(X)$?

5. Convex Hulls

The main result in this section is Theorem 5E, which completes a partial result found in [12]. We approach this by means of Proposition 5D; the core of the argument is in Lemma 5C. The remainder of the section is made up of relatively straightforward corollaries.

5A. Notation. For any sets A, X and $S \subseteq A \times X$, we shall write

$$\pi_1[S] = \{x : \exists t, (x,t) \in S\},$$
$$S(x) = \{t : (x,t) \in S\},$$
$$S^{-1}(t) = \{x : (x,t) \in S\}.$$

If Σ and $\mathfrak{B}$ are σ-subalgebras of subsets of A, X respectively, $\Sigma \hat{\otimes}_\sigma \mathfrak{B}$ will be the σ-algebra of subsets of $A \times X$ generated by $\{E \times F : E \in \Sigma, F \in \mathfrak{B}\}$.

5B. Lemma. *Let (A, Σ, μ) be a complete probability space and X a compact metric space. Let $\mathfrak{B} = \mathfrak{B}(X)$ be the algebra of Borel sets of X. Then $\pi_1[S] \in \Sigma$ for every $S \in \Sigma \hat{\otimes}_\sigma \mathfrak{B}$.*

Proof. If $\mathcal{Q}$ is the class of sets

$$\{E \times F : E \in \Sigma, F \subseteq X \text{ closed}\},$$

then S is $\mathcal{Q}$-Souslin, because the class of $\mathcal{Q}$-Souslin sets is closed under countable unions and intersections and contains the complements of members of $\mathcal{Q}$ (because X is metrizable). So S is expressible in the form

$$\bigcup \left\{ \bigcap_{n \in \mathbf{N}} E_{\varphi|n} \times F_{\varphi|n} : \varphi \in \mathbf{N}^{\mathbf{N}} \right\},$$

where $\varphi|n = (\varphi(0), \ldots, \varphi(n))$ and $E_{\varphi|n} \in \Sigma$, $F_{\varphi|n}$ is closed in X for every φ and n. We can suppose also that the system $\langle E_{\varphi|n} \times F_{\varphi|n} \rangle_{\varphi, n}$ is regular in the sense that

$E_{\varphi|n+1} \subseteq E_{\varphi|n}$, $F_{\varphi|n+1} \subseteq F_{\varphi|n}$ for every φ, n. Now

$$\pi_1[S] = \bigcup \left\{ \pi_1\left[\bigcap_{n\in\mathbf{N}} E_{\varphi|n} \times F_{\varphi|n} \right] : \varphi \in \mathbf{N}^{\mathbf{N}} \right\}$$
$$= \bigcup \left\{ \bigcap_{n\in\mathbf{N}} \pi_1\left[E_{\varphi|n} \times F_{\varphi|n} \right] : \varphi \in \mathbf{N}^{\mathbf{N}} \right\}$$

because X is compact. So $\pi_1[S]$ is Σ-Souslin and belongs to Σ because (A, Σ, μ) is complete (see [9], p. 95).

5C. Lemma. *Let (A, Σ, μ) be a complete probability space and (X, ρ) a compact metric space. Let S and T be two subsets of $A \times X$ such that $S^{-1}(t)$ and $T^{-1}(t)$ belong to Σ for every $t \in X$ and*

for every $x \in A$ and every non-empty closed set $F \subseteq X$, at least one of $\overline{F \cap S(x)}$, $\overline{F \cap T(x)}$ is not equal to F.

Then, for any $\delta > 0$ and any non-empty open set $U \subseteq X$, there is a non-empty open set $V \subseteq U$ such that

$$\mu S^{-1}(s) + \mu T^{-1}(t) \leqslant 1 + 3\delta \qquad \forall s, t \in V.$$

Proof. If $X = \varnothing$, there is nothing to prove; so we shall suppose $X \neq \varnothing$. Fix a sequence $\langle V_k \rangle_{k\in\mathbf{N}}$ of non-empty open sets in X which runs over a base for the topology of X.

(A) We begin by supposing that S, T both belong to $\Sigma \hat{\otimes}_\sigma \mathfrak{B}(X)$, where $\mathfrak{B}(X)$ is the algebra of Borel sets in X.

(a) Define inductively, for ordinals ξ less than the first uncountable ordinal Ω, sets $\Psi_\xi \subseteq A \times X$ as follows:

$$\Psi_0 = A \times X;$$

given Ψ_ξ, where ξ is even (i.e., of the form $\eta + 2k$, where η is a limit ordinal and $k \in \mathbf{N}$),

$$\Psi_{\xi+1} = \left\{ (x,t) : x \in A, t \in \overline{S(x) \cap \Psi_\xi(x)} \right\},$$
$$\Psi_{\xi+2} = \left\{ (x,t) : x \in A, t \in \overline{T(x) \cap \Psi_{\xi+1}(x)} \right\};$$

and, for limit ordinals $\xi < \Omega$,

$$\Psi_\xi = \bigcap_{\eta < \xi} \Psi_\eta.$$

We see at once that $\Psi_\xi(x)$ is closed for every $x\in A$, $\xi<\Omega$; that $\Psi_\xi\subseteq\Psi_\eta$ if $\eta\leqslant\xi<\Omega$; and that $\Psi_{\xi+2}(x)\subset\Psi_\xi(x)$ if $\Psi_\xi(x)\neq\varnothing$, by the hypothesis on S and T.

(b) We induce on ξ to show that $\Psi_\xi\in\Sigma\hat{\otimes}_\sigma\mathfrak{B}$ for every ξ. For $\xi=0$, this is trivial; and the inductive step to countable limit ordinals is also trivial. For successor ordinals, suppose that $\Psi_\xi\in\Sigma\hat{\otimes}_\sigma\mathfrak{B}$, where ξ is even. Then

$$\begin{aligned}\Psi_{\xi+1} &= \left\{(x,t): t\in\overline{S(x)\cap\Psi_\xi(x)}\right\}\\ &= \bigcap_{k\in\mathbf{N}}\left\{(x,t):\text{either } t\notin V_k \text{ or } V_k\cap S(x)\cap\Psi_\xi(x)\neq\varnothing\right\}\\ &= \bigcap_{k\in\mathbf{N}}\left[\left[A\times(X\setminus V_k)\right]\cup\left(\pi_1\left[(A\times V_k)\cap S\cap\Psi_\xi\right]\times X\right)\right],\end{aligned}$$

which belongs to $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$ by Lemma 5B and the inductive hypothesis (as we are assuming that $S\in\Sigma\hat{\otimes}_\sigma\mathfrak{B}$). Similarly,

$$\begin{aligned}\Psi_{\xi+2} &= \bigcap_{k\in\mathbf{N}}\left[\left[A\times(X\setminus V_k)\right]\cup\left(\pi_1\left[(A\times V_k)\cap T\cap\Psi_{\xi+1}\right]\times X\right)\right]\\ &\in\Sigma\hat{\otimes}_\sigma\mathfrak{B}.\end{aligned}$$

(c) For $k\in\mathbf{N}$, $\xi<\Omega$ write

$$E_{k\xi} = \pi_1\left[(A\times V_k)\cap S\cap\Psi_\xi\right].$$

Then, for any fixed k, $\langle E_{k\xi}\rangle_{\xi<\Omega}$ is a decreasing family in Σ. Consequently there is an even $\eta<\Omega$ such that $\mu E_{k\xi}=\mu E_{k\eta}$ for every $\xi\geqslant\eta$, $k\in\mathbf{N}$. Set

$$A_1 = A\setminus\bigcup_{k\in\mathbf{N}}(E_{k\eta}\setminus E_{k,\eta+2}),$$

so that $\mu(A\setminus A_1)=0$. If $x\in A_1$, then

$$\left\{k: x\in E_{k,\eta+2}\right\} = \left\{k: x\in E_{k\eta}\right\},$$

so that

$$\begin{aligned}\Psi_{\eta+1}(x) &= \left\{t:\forall k\in\mathbf{N}, \text{ either } t\notin V_k \text{ or } x\in E_{k\eta}\right\}\\ &= \Psi_{\eta+3}(x).\end{aligned}$$

But this can happen only if $\Psi_{\eta+1}(x)=\varnothing$. Thus we have found a countable ordinal $\zeta=\eta+1$ and a conegligible set $A_1\subseteq A$ such that $\Psi_\zeta(x)=\varnothing$ for every $x\in A_1$.

(d) For $n \in \mathbf{N}$, $\xi < \Omega$ set

$$\Phi_{n\xi} = \{(x,t) : (x,t) \in \Psi_\xi,\ \rho(t, \Psi_{\xi+1}(x)) \geq 2^{-n}\}.$$

To see that $\Phi_{n\xi} \in \Sigma \hat{\otimes}_\sigma \mathfrak{B}$, note that if $\langle t_k \rangle_{k \in \mathbf{N}}$ runs over a dense set in X,

$$\Phi_{n\xi} = \Psi_\xi \setminus \bigcup \{R(\alpha, \beta, k) : \alpha, \beta \in \mathbf{Q},\ \alpha + \beta < 2^{-n},\ k \in \mathbf{N}\},$$

where

$$\begin{aligned} R(\alpha,\beta,k) &= \{(x,t) : \rho(t,t_k) \leq \beta,\ N_\alpha(t_k) \cap \Psi_{\xi+1}(x) \neq \varnothing\} \\ &= \pi_1([A \times N_\alpha(t_k)] \cap \Psi_{\xi+1}) \times N_\beta(t_k), \end{aligned}$$

writing $N_\alpha(t_k) = \{t : \rho(t,t_k) \leq \alpha\}$. Of course every $\Phi_{n\xi}(x)$ is closed. Moreover, if $\eta < \xi$, then $\rho(\Phi_{n\xi}(x), \Phi_{n\eta}(x)) \geq 2^{-n}$ for every n and x (following the convention that $\rho(\varnothing, F) = \infty$ for every set F). As every $\Psi_{\xi+1}(x)$ is closed, we have

$$\bigcup_{n \in \mathbf{N}} \Phi_{n\xi} = \Psi_\xi \setminus \Psi_{\xi+1} \qquad \forall \xi < \Omega,$$

and

$$\bigcup_{n \in \mathbf{N}, \eta < \xi} \Phi_{n\eta} = (A \times X) \setminus \Psi_\xi \qquad \forall \xi < \Omega.$$

Observe also that $\Phi_{n\xi} \subseteq \Phi_{n+1,\xi}$ for every n, ξ.

(e) Recall the definition of ζ and A_1 from (c) above, and set

$$\Phi_n = \bigcup_{\xi < \zeta} \Phi_{n\xi} \in \Sigma \hat{\otimes}_\sigma \mathfrak{B};$$

define h on $A \times X$ by writing

$$\begin{aligned} h(x,t) &= 1 \qquad \text{if} \quad (x,t) \in \Psi_\xi \setminus \Psi_{\xi+1}, \quad \text{where } \xi \text{ is odd, } \xi \leq \zeta, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then h is $\Sigma \hat{\otimes}_\sigma \mathfrak{B}$-measurable. Now we know that, if $x \in A_1$ and $t \in X$, then $t \notin \Psi_\zeta(x)$, so there is some $\xi < \zeta$ such that $t \in \Psi_\xi(x) \setminus \Psi_{\xi+1}(x)$. If ξ is even, then $h(x,t) = 0$ and $t \notin \overline{S(x) \cap \Psi_\xi(x)}$, so $(x,t) \notin S$; if ξ is odd, then $h(x,t) = 1$ and $t \notin \overline{T(x) \cap \Psi_\xi(x)}$, so $(x,t) \notin T$. Turning these around, we see that, for any $x \in A_1$

and $t\in X$,

$$(\chi S)(x,t) \leqslant h(x,t), \qquad (\chi T)(x,t) \leqslant 1-h(x,t).$$

(f) For any $x\in A$, $n\in\mathbf{N}$, $\xi<\Omega$ we see that $h(x,t)$ is constant for $t\in\Phi_{n\xi}(x)$. As the $\Phi_{n\xi}(x)$, for fixed n and x, are isolated, the restriction of h_x to $\Phi_n(x)$ is continuous, where $h_x(t)=h(x,t)$. Let B be the unit ball of $C(X)$. We define correspondences $\Theta_n\subseteq A\times B$ by

$$\Theta_n = \{(x,z): x\in A,\, z\in B,\, z(t)=h(x,t)\ \forall t\in\Phi_n(x)\}.$$

By Tietze's theorem, $\Theta_n(x)$ is never empty, and clearly it is closed (here, for once, we are giving B the uniform norm topology, so that B is a Polish space). To show that Θ_n is a measurable correspondence, it is enough to show that

$$\{x: d(z,\Theta_n(x))\leqslant\varepsilon\}$$

is measurable for each $z\in B$, $\varepsilon>0$, where d is the metric of B. But

$$\begin{aligned}\{x: d(z,\Theta_n(x))\leqslant\varepsilon\} &= \{x: |z(t)-h(x,t)|\leqslant\varepsilon\ \forall t\in\Phi_n(x)\}\\ &= A\setminus\pi_1[\Phi_n\cap\{(x,t): |z(t)-h(x,t)|>\varepsilon\}]\in\Sigma\end{aligned}$$

because h is $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$-measurable and $\Phi_n\in\Sigma\hat{\otimes}_\sigma\mathfrak{B}$.

Consequently, by the selection theorem of [8], there exist measurable functions $\theta_n: A\to B$ such that $\theta_n(x)\in\Theta_n(x)$ for every $x\in A$. Set $f_n(x,t)=\theta_n(x)(t)$. Then each f_n is measurable in the first variable and continuous in the second, and $|f_n(x,t)|\leqslant 1$ for every x,t. Moreover, f_n and h agree on Φ_n. Since $\langle\Phi_n(x)\rangle_{n\in\mathbf{N}}\uparrow X$ for $x\in A_1$, we have

$$h(x,t) = \lim_{n\in\mathbf{N}} f_n(x,t) \qquad \forall x\in A_1,\ t\in X.$$

(g) If we now set

$$z_n(t) = \int f_n(x,t)\mu(dx),$$

then $z_n\in C(X)$ (as f_n is continuous in the second variable and X is metrizable) and

$$\lim_{n\in\mathbf{N}} z_n(t) \text{ exists } = \int h(x,t)\mu(dx) \qquad \forall t\in X,$$

because, for any $t\in X$, $h(x,t)=\lim_{n\in\mathbf{N}} f_n(x,t)$ for almost all x.

(h) Returning to the set U of the enunciation of the lemma, we see that

$$U = \bigcup_{n\in\mathbf{N}} \{t : t\in U,\ |z_m(t)-z_n(t)| \leqslant \delta\ \forall m \geqslant n\};$$

by Baire's theorem, there is an $n\in\mathbf{N}$ such that

$$G = \operatorname{int}\{t : t\in U,\ |z_m(t)-z_n(t)| \leqslant \delta\ \forall m \geqslant n\}$$

is not empty. Let $V\subseteq G$ be a non-empty open set such that $|z_n(s)-z_n(t)|\leqslant\delta$ for every $s,t\in V$, so that $|z_m(s)-z_n(t)|\leqslant 3\delta$ for every $s,t\in V$ and $m\geqslant n$.

Since, for any $x\in A_1$ and $s,t\in V$,

$$(\chi S)(x,s) + (\chi T)(x,t) \leqslant h(x,s) + 1 - h(x,t),$$

we have

$$\begin{aligned}\mu S^{-1}(s) + \mu T^{-1}(t) &\leqslant 1 + \int h(x,s)\mu(dx) - \int h(x,t)\mu(dx)\\ &= 1 + \lim_{m\in\mathbf{N}} [z_m(s) - z_m(t)]\\ &\leqslant 1+3\delta,\end{aligned}$$

by the choice of V.

(B) This completes the argument when S and T belong to $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$. To cover the general case, we argue by contradiction. Suppose, if possible, that no such V can be found. Let $I=\{k: V_k\cap U\neq\varnothing\}$. Then we can choose, for each $k\in I$, points $s_k, t_k\in V_k\cap U$ such that $\mu S^{-1}(s_k)+\mu T^{-1}(t_k)>1+3\delta$. If we set

$$S_0 = \bigcup_{k\in I} S^{-1}(s_k)\times\{s_k\}, \qquad T_0 = \bigcup_{k\in I} T^{-1}(t_k)\times\{t_k\},$$

then we see that S_0 and T_0 belong to $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$; that because $S_0\subseteq S$ and $T_0\subseteq T$, there is no non-empty closed $F\subseteq X$ and $x\in A$ such that $F=\overline{F\cap S_0(x)}=\overline{F\cap T_0(x)}$; and that if V is any non-empty open set in U,

$$\sup_{s\in V}\mu S_0^{-1}(s) + \sup_{t\in V}\mu T_0^{-1}(t) > 1+3\delta,$$

contradicting the result of part (A).

Remark. It will be seen that for each $x\in A$ taken separately, the construction of $\langle\Psi_\xi(x)\rangle_{\xi<\Omega}, \langle\Phi_{n\xi}(x)\rangle_{n\in\mathbf{N},\ \xi<\Omega}$ above forms one of the methods of finding a sequence $\langle f_n(x,\cdot)\rangle_{n\in\mathbf{N}}$ of continuous functions converging to 1 on

$S(x)$ and 0 on $T(x)$. The essential idea of this proof is to do this for all x simultaneously in such a way that the f_n will be measurable in the first variable.

5D. Proposition. *Let (A,Σ,μ) be a complete probability space and X a complete metric space. Let $f:A\times X\to\mathbf{R}$ be a bounded function, measurable in the first variable and of the first Baire class in the second. Then, writing*

$$z(t)=\int f(x,t)\mu(dx),$$

we shall have $z\in\boldsymbol{B}_1(X)$.

Proof. By Proposition 1E, it is enough to show that $z|_K\in\boldsymbol{B}_1(K)=\boldsymbol{B}_r^{\S}(K)$ for every compact $K\subseteq X$; of course, this is the same as showing that $z|_K\in\boldsymbol{B}^{\S}(K)$ for every compact $K\subseteq X$; considering the restriction of the function f to $A\times K$, we see that it is enough to prove that $z\in\boldsymbol{B}^{\S}(X)$ under the additional hypothesis that X is compact. By Lemma 1C(a), it is enough to show that for each $\varepsilon>0$ and non-empty open set $U\subseteq X$, there is a non-empty open set $V\subseteq U$ such that $\operatorname{diam}(z[V])\leqslant\varepsilon$. Moreover, as f is bounded and $\boldsymbol{B}_1(X)$ is a linear space, we can confine ourselves to the case in which $0\leqslant f(x,t)\leqslant 1$ for every x,t.

Let $n\in\mathbf{N}$ be so large that $3n+1\leqslant\varepsilon n^2$. Set

$$S_r=\left\{(x,t):f(x,t)\leqslant\frac{r}{n}\right\},\qquad T_r=\left\{(x,t):f(x,t)\geqslant\frac{r}{n}\right\}.$$

Then, for each r, S_r and T_{r+1} satisfy the conditions of Lemma 5C, because, for each $x\in A$, the function $t\mapsto f(x,t):X\to\mathbf{R}$ belongs to $\boldsymbol{B}_1(X)=\boldsymbol{B}_r^{\S}(X)$. So we may find inductively non-empty open sets V_r such that $V_0=U$, $V_{r+1}\subseteq V_r$, and

$$\mu S_r^{-1}(s)+\mu T_{r+1}^{-1}(t)\leqslant 1+\frac{1}{n}\qquad\forall s,t\in V_{r+1},$$

for $0\leqslant r\leqslant n$. Now suppose that $s,t\in V=V_{n+1}$. Then

$$\sum_{r\leqslant n}\frac{1}{n}\mu T_{r+1}^{-1}(t)\geqslant\int\left(f(x,t)-\frac{1}{n}\right)\mu(dx)=z(t)-\frac{1}{n},$$

while also

$$\sum_{r\leqslant n}\frac{1}{n}\left[1-\mu S_r^{-1}(s)\right]\leqslant z(s)+\frac{1}{n}.$$

So

$$z(t)-z(s) \leqslant \sum_{r \leqslant n} \frac{1}{n} \mu T_{r+1}^{-1}(t)+\frac{1}{n}-\sum_{r \leqslant n} \frac{1}{n}\left[1-\mu S_r^{-1}(s)\right]+\frac{1}{n}$$
$$=\frac{2}{n}+\frac{1}{n} \sum_{r \leqslant n}\left[\mu T_{r+1}^{-1}(t)+\mu S_r^{-1}(s)-1\right]$$
$$\leqslant \frac{2}{n}+\frac{1}{n}(n+1) \frac{1}{n} \leqslant \varepsilon .$$

This proves the result.

5E. Theorem. *Let X be a complete metric space, $A \subseteq \boldsymbol{B}_1(X)$ a compact uniformly bounded set. Then its convex hull $\Gamma(A)$ is relatively compact in $\boldsymbol{B}_1(X)$.*

Proof. The closed convex hull $\overline{\Gamma(A)}$ of A in $\mathbf{R}^X$ is surely compact, so it is enough to show that $\overline{\Gamma(A)} \subseteq \boldsymbol{B}_1(X)$. Let $z \in \overline{\Gamma(A)}$. As A is a compact set in the locally convex Hausdorff space $\mathbf{R}^X$, there is a Radon measure μ on A such that

$$fz=\int_A f(x) \mu(dx)$$

for every $f \in (\mathbf{R}^X)'$ [1, Chapter IV, §7, No. 1, Proposition 1]. In particular, $z(t)=\int_A x(t) \mu(dx)$ for every $t \in X$. But if we consider the function $h: A \times X \rightarrow \mathbf{R}$ given by

$$h(x, t)=x(t) \qquad \forall x \in A,\ t \in X,$$

this satisfies the conditions of Proposition 5D. So $z \in \boldsymbol{B}_1(X)$ as required.

5F. Example. If A is not uniformly bounded, the result fails; for instance, take $X=[0,1]$, and enumerate $\mathbf{Q} \cap[0,1]$ as $\langle q_n\rangle_{n \in \mathbf{N}}$. Set

$$A=\left\{2^{n+1} \chi\{q_n\}: n \in \mathbf{N}\right\} \cup\{0\}.$$

Then A is a compact set in $\boldsymbol{B}_1(X)$, but $\overline{\Gamma(A)}$, taken in $\mathbf{R}^X$, contains $\chi(\mathbf{Q} \cap[0,1])$, so that $\Gamma(A)$ is not relatively compact in $\boldsymbol{B}_1(X)$.

However, we do have two results passing around this obstacle.

5G. Corollary. *Let X be a complete metric space, and $A \subseteq \boldsymbol{C}(X)$ any set which is relatively compact in $\boldsymbol{B}_1(X)$. Then $\Gamma(A)$ is relatively compact in $\boldsymbol{B}_1(X)$.*

Proof. We know that A is pointwise bounded, so that

$$w(t) = 1 + \sup_{x \in A} |x(t)|$$

exists for each $t \in X$ (ignoring the trivial case $A = \varnothing$). As $A \subseteq \boldsymbol{C}(X)$, w is semicontinuous, and $w \in \boldsymbol{B}_1(X)$ [Corollary 1D(b)]. Now the map $x \mapsto x/w$ is a linear $\mathfrak{T}_p$-homeomorphism of $\boldsymbol{B}_1$ onto itself ([9], §31, Part VI, Theorem 2), so $D = \{x/w : x \in A\}$ is relatively compact in $\boldsymbol{B}_1$. But D is uniformly bounded by the constant 1, so we can apply Theorem 5E to see that $\Gamma(D)$ is relatively compact in $\boldsymbol{B}_1$; multiplying back up by w, we see that $\Gamma(A)$ is relatively compact in $\boldsymbol{B}_1$.

Remark. Using Proposition 1E, it is possible to derive this corollary, and therefore 5I and 5J below, from the special case treated in [12] (where X is supposed compact and A uniformly bounded).

5H. Proposition.

(a) *If X is a complete metric space and $A \subseteq \boldsymbol{B}_1(X)$ is a compact set, then $\Gamma(A)$ is relatively compact in $\boldsymbol{B}_2(X)$.*
(b) *If X is Polish and $A \subseteq \boldsymbol{B}_1(X)$ is a compact set, then $\overline{\Gamma(A)}$, taken in $\mathbf{R}^X$, is angelic.*

Proof. (a): For each $n \in \mathbf{N}$, let $\varphi_n : \boldsymbol{B}_1(X) \to \boldsymbol{B}_1(X)$ be given by

$$(\varphi_n x)(t) = \max(-n, \min(n, x(t))).$$

Then φ_n is continuous, so $\varphi_n[A]$ is compact and uniformly bounded.

As in Theorem 5E, we have to show that if μ is a Radon probability measure on A, then $z \in \boldsymbol{B}_2(X)$, where

$$z(t) = \int_A x(t) \mu(dx) \qquad \forall t \in X,$$

Now $\varphi_n : A \to \varphi_n[A]$ defines a Radon probability measure $\mu_n = \mu \varphi_n^{-1}$ on $\varphi_n[A]$. Writing

$$z_n(t) = \int_{\varphi_n[A]} x(t) \mu_n(dx),$$

we shall have $z_n \in \overline{\Gamma(\varphi_n[A])} \subseteq \boldsymbol{B}_1(X)$ by Theorem 5E. But

$$z_n(t) = \int_A (\varphi_n x)(t) \mu(dx)$$

by the definition of $\mu_n = \mu\varphi_n^{-1}$. As, for each $t \in X$, $x(t) = \lim_{n\in\mathbf{N}}(\varphi_n x)(t)$ for every $x \in A$, and as $\sup_{x\in A}|x(t)| < \infty$, we shall have $z(t) = \lim_{n\in\mathbf{N}} z_n(t)$ by Lebesgue's theorem. As t is arbitrary, $z = \lim_{n\in\mathbf{N}} z_n$ in $\mathbf{R}^X$, and $z \in \boldsymbol{B}_2(X)$.

(b): We continue the argument, now adding the assumption that X is separable. As $\overline{\Gamma(A)}$ is compact, it will be angelic if whenever $B \subseteq \overline{\Gamma(A)}$ and $z \in \bar{B}$, there is a sequence in B converging to z. Set $X_n = \{t : t \in X, |x(t)| \leqslant n \ \forall x \in A\}$, $A_n = \overline{\Gamma(\varphi_n[A])} \subseteq \boldsymbol{B}_1(X)$, and

$$B_n = \{ y : y \in A_n, \exists w \in B, y_n(t) = w(t) \ \forall t \in X_n \}$$

for each $n \in \mathbf{N}$.

We observe first that all the B_n are adequately large. Let w be any member of $\overline{\Gamma(A)}$. Let $\varepsilon > 0$, and let $F \subseteq X_n$ be finite. Then there exist $x_0, \ldots, x_k \in A$ and $\alpha_i \geqslant 0$ such that $\sum_{i\leqslant k}\alpha_i = 1$ and

$$\left| w(t) - \sum_{i\leqslant k} \alpha_i x_i(t) \right| \leqslant \varepsilon \qquad \forall t \in F.$$

As $F \subseteq X_n$,

$$\left| w(t) - \sum_{i\leqslant k} \alpha_i (\varphi_n x_i)(t) \right| \leqslant \varepsilon \qquad \forall t \in F.$$

Thus w is approximated by members of A_n uniformly on finite subsets of X_n. As A_n is compact, there is a $y \in A_n$ such that $y(t) = w(t)$ for every $t \in X_n$. In particular, if $w \in B$, there is a $y \in B_n$ agreeing with w on X_n.

Consequently z, which can be approximated uniformly on finite subsets of X by members of B, can be approximated uniformly on finite subsets of X_n by members of B_n; as $\bar{B}_n$ is compact, there is a $z_n \in \bar{B}_n$ agreeing with z on X_n. Now $\bar{B}_n$ is angelic (this is where we use the hypothesis that X is Polish), so there is a sequence $\langle y_{nm} \rangle_{m\in\mathbf{N}}$ in B_n such that $z_n = \lim_{m\in\mathbf{N}} y_{nm}$. If we choose $w_{nm} \in B$ agreeing with y_{nm} on X_n, we shall have

$$z(t) = \lim_{m\in\mathbf{N}} w_{nm}(t) \qquad \forall t \in X_n.$$

Since $\langle X_n \rangle_{n\in\mathbf{N}} \uparrow X$, we see that $z \in \overline{\{w_{nm} : n, m \in \mathbf{N}\}}$ in $\mathbf{R}^X$. But $C = \{w_{nm} : n, m \in \mathbf{N}\}$ is a countable relatively compact set in $\boldsymbol{B}(X)$, so $\bar{C}$ is angelic, by Theorem 4D (v)⇒(vi), and z is the limit of a sequence in C, which is the required sequence in B converging to z.

5I. We now show how Theorem 5E also gives easy extensions to Theorems 2F and 4D.

PROPOSITION. *If X is any Hausdorff space and $A \subseteq \boldsymbol{C}(X)$ is a pointwise bounded set, then $\Gamma(A)$ satisfies the conditions of Theorem 2F iff A does.*

Proof. Of course, if $\Gamma(A)$ satisfies any of the conditions of 2F, then A satisfies the same condition.

For the converse, we argue as in part (e) of the proof of 2F. Suppose that A satisfies condition 2F(vi). Let $\langle x_n \rangle_{n \in \mathbf{N}}$ be any sequence in $\Gamma(A)$, and $K \subseteq X$ a compact set. Then there is a sequence $\langle z_n \rangle_{n \in \mathbf{N}}$ in A such that every x_n belongs to $\Gamma\{z_n : n \in \mathbf{N}\}$; define $\varphi : K \to \mathbf{R}^{\mathbf{N}}$ by $(\varphi t)(n) = z_n(t)$. Set $L = \varphi[K]$ and let B be the set of coordinate functionals on L.

As before, we find that B is relatively compact in $\boldsymbol{B}_1(L)$. By Corollary 5G, $\Gamma(B)$ is relatively compact in $\boldsymbol{B}_1(L)$, and therefore relatively sequentially compact in $\mathbf{R}^L$, so $\Gamma(A_K)$ is relatively sequentially compact in $\mathbf{R}^K$, and $\langle x_n|_K \rangle_{n \in \mathbf{N}}$ has a convergent subsequence. But this has proved that $\Gamma(A)$ satisfies 2F(ii).

5J. Proposition. *Let X be a K-analytic space and $A \subseteq \boldsymbol{Ka}(X)$ a countable pointwise bounded set. Then $\Gamma(A)$ satisfies any one of the conditions of Theorem 4D iff A does; in which case both A and $\Gamma(A)$ satisfy all the conditions of Theorem 4D.*

Proof. If $\Gamma(A)$ satisfies any one of the conditions of 4D, then A satisfies the same one, and therefore all the others. If A satisfies the conditions of 4D, consider the set $C \subseteq \boldsymbol{C}(Z)$ formed in the course of part (c) of the proof of 4D. We know that C is relatively compact in $\boldsymbol{B}_1(Z)$, so that $\Gamma(C)$ also is (Corollary 5G again), and $\overline{\Gamma(C)}$ is angelic, by Theorem 3F. But $\overline{\Gamma(C)}$ is homeomorphic to $\overline{\Gamma(A)}$ (Lemma 2E), so $\overline{\Gamma(A)}$ is angelic. It is now elementary that $\Gamma(A)$ satisfies all the conditions (i)–(viii) of 4D, even though it be uncountable.

5K. Remarks and Problems. We now know the following, for a complete metric space X.

(a) If $A \subseteq \boldsymbol{C}(X)$ is compact and uniformly bounded, then $\overline{\Gamma(A)} \subseteq \boldsymbol{C}(X)$. (In fact, by Krein's theorem, this is true for any compact Hausdorff space X, and therefore for any Hausdorff k-space; see [7], §24.5.)

(b) If $A \subseteq \boldsymbol{C}(X)$ is compact, then $\overline{\Gamma(A)} \subseteq \boldsymbol{B}_1(X)$ (Corollary 5G).

(c) If $A \subseteq \boldsymbol{B}_1(X)$ is compact and uniformly bounded, then $\overline{\Gamma(A)} \subseteq \boldsymbol{B}_1(X)$ (Theorem 5E).

(d) If $A \subseteq \boldsymbol{B}_1(X)$ is compact, then $\overline{\Gamma(A)} \subseteq \boldsymbol{B}_2(X)$ (Proposition 5H).

While we see no reason to suppose that this series can be extended, it would be agreeable to know. A more important question seems to be the following:

If X is Polish and $A \subseteq \boldsymbol{B}(X)$ is compact, does it follow that $\Gamma(A)$ is relatively compact in $\boldsymbol{B}(X)$?

6. Countably Compact Sets in $\hat{\boldsymbol{B}}(X)$.

We end this paper by examining some results concerning $\hat{\boldsymbol{B}}(X)$, the space of functions with the weak Baire property, where X is a Polish space. They are of the same type as those given in [6] for the space of measurable function on a Radon measure space. Our first lemma is a refined version of 2B above.

6A. Lemma. *Let X be a Polish space and $\langle x_n \rangle_{n\in\mathbf{N}}$ a sequence of $\{0,1\}$-valued functions in $\hat{\boldsymbol{B}}(X)$ such that every subsequence has a cluster point in $\hat{\boldsymbol{B}}(X)$. Set $E_n = x_n^{-1}[\{1\}]$, $F_n = X \setminus E_n$. Then for every non-empty open set $G \subseteq X$ there is a non-empty open set $H \subseteq G$ such that one of $\{n : H \cap E_n$* meager$\}$, $\{n : H \cap F_n$ meager$\}$ *is infinite.*

Proof. Suppose, if possible, otherwise. In this case, for every non-meager set $H \subseteq G$ (whether open or not), $\{n : H \cap E_n$ meager$\}$ and $\{n : H \cap F_n$ meager$\}$ are finite. (For if H is not open, we can find an open H_1 such that, for any set E with the Baire property, $E \cap H_1$ is meager iff $E \cap H$ is meager.) Enumerate a base for the open sets of G as $\langle V_k \rangle_{k\in\mathbf{N}}$, with $V_0 = G$ and no V_k empty. Construct a strictly increasing sequence $\langle n(r) \rangle_{r\in\mathbf{N}}$ such that

(i) $G \cap E_{n(0)}$, $G \cap F_{n(0)}$ are non-meager;

(ii) if (P,Q) is a partition of $\{0,\ldots,r\}$ and $k \leqslant r$ is such that $V_k \cap C_{PQ}$ is not meager, where

$$C_{PQ} = \bigcap_{i\in P} E_{n(i)} \cap \bigcap_{i\in Q} F_{n(i)},$$

then $V_k \cap C_{PQ} \cap E_{n(r+1)}$, $V_k \cap C_{PQ} \cap F_{n(r+1)}$ are not meager. (Any $n(r+1)$ large enough will do.)

It will follow that if $P' \supseteq P$ and $Q' \supseteq Q$ and $V_k \cap C_Q P$ is not meager, where (P,Q) is a partition of $\{0,\ldots,r\}$ and $k \leqslant r$, then $V_k \cap C_{P'Q'}$ will not be meager.

Let x be any cluster point in $\hat{\boldsymbol{B}}(X)$ of $\langle x_{n(r)} \rangle_{r\in\mathbf{N}}$. Let $Y \subseteq X$ be a dense G_δ set such that x and every $x_{n(r)}$ have continuous restrictions to Y. Define $\varphi : Y \to \mathcal{P}\mathbf{N}$ by writing $\varphi t = \{r : t \in E_{n(r)}\}$. Then φ is continuous. There is a non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$ such that $x = \lim_{r\to\mathcal{F}} x_{n(r)}$; now $Y_1 = \{t : t \in Y, x(t) = 1\} = \varphi^{-1}[\mathcal{F}]$. Since Y_1 and $Y \setminus Y_1 = Y_0$ are open-and-closed sets in the dense G_δ set Y in the Polish space X, they are Polish, and $\varphi[Y_1]$ and $\varphi[Y_0]$ are

Souslin; as $\varphi[Y_1]\subseteq\mathcal{F}$ and $\varphi[Y_0]\subseteq\mathcal{P}\mathbf{N}\setminus\mathcal{F}$, Lemma 2D tells us that both are meager, and $\varphi[Y]$ is meager in $\mathcal{P}\mathbf{N}$.

Let $\langle H_r\rangle_{r\in\mathbf{N}}$ be a sequence of nowhere dense closed sets in $\mathcal{P}\mathbf{N}$ such that $\varphi[Y]\subseteq\bigcup_{r\in\mathbf{N}}H_r$. Let $Y_r=\varphi^{-1}[H_r]$, so that $Y=\bigcup_{r\in\mathbf{N}}Y_r$. As Y is a dense G_δ set and G is a non-empty open set, $Y\cap G$ is not meager, and there is some $r\in\mathbf{N}$ such that $Y_r\cap G$ is not meager. Let k be such that $V_k\setminus(Y_r\cap G)$ is meager. As

$$X=\bigcup\{C_{PQ}:(P,Q)\text{ a partition of }\{0,\ldots,k\}\},$$

there is some partition (P,Q) of $\{0,\ldots,k\}$ such that $V_k\cap C_{PQ}$ is not meager.

Now H_r is supposed to be nowhere dense in $\mathcal{P}\mathbf{N}$, so there must be some $m\geqslant k$ and a partition (P',Q') of $\{0,\ldots,m\}$ such that $P'\supseteq P$, $Q'\supseteq Q$ and

$$H_r\cap\{I:I\subseteq\mathbf{N},I\cap\{0,\ldots,m\}=P'\}=\varnothing.$$

It follows that $Y_r\cap C_{P'Q'}=\varnothing$, for this is the inverse image of the last under φ. But we now have $V_k\setminus Y_r$ meager, so $V_k\cap C_{P'Q'}$ must be meager, contradicting the choice of the $n(i)$. This proves the result.

6B. Theorem. *Let X be a Polish space and $A\subseteq\hat{\mathbf{B}}(X)$ a relatively countably compact set. If $\langle x_n\rangle_{n\in\mathbf{N}}$ is a sequence in A with a cluster point $x\in\mathbf{R}^X$, then there is a subsequence of $\langle x_n\rangle_{n\in\mathbf{N}}$ converging to x almost everywhere (i.e., except on a meager set).*

Proof. (a) We suppose first that A consists entirely of characteristic functions. Note that if $I\subseteq\mathbf{N}$ is an infinite set such that x is a cluster point of $\langle x_n\rangle_{n\in I}$, and G is an open set such that x_n is essentially constant on G for each $n\in I$, then x is essentially constant on G.

Enumerate a base for the topology of X as $\langle V_k\rangle_{k\in\mathbf{N}}$. (We pass over the case in which X is finite.) Set $I_0=\mathbf{N}$, and defne I_k, for $k\geqslant1$, inductively by saying that I_{k+1} will always be one of the three sets

$$I'_k=\{n:n\in I_k,\ x_n=1\text{ essentially everywhere on }V_k\},$$
$$I''_k=\{n:n\in I_k,\ x_n=0\text{ essentially everywhere on }V_k\},$$
$$I_k\setminus(I'_k\cup I''_k),$$

and that x must always be a cluster point of $\langle x_n\rangle_{n\in I_{k+1}}$. Let $I\subseteq\mathbf{N}$ be any infinite set such that $I\setminus I_k$ is finite for every $k\in\mathbf{N}$. Now, if it were the case that for some non-empty open set $G\subseteq X$, $I_{k+1}=I_k\setminus(I'_k\cup I''_k)$ whenever $V_k\subseteq G$ and $V_k\neq\varnothing$, then $\langle x_n\rangle_{n\in I}$ would contradict Lemma 6A, since x_n would take both values essentially on V_k for every $n\in I\cap I_{k+1}$. So, for every non-empty open

$G \subseteq X$, there is a non-empty $V_k \subseteq G$ such that all the x_n, for $n \in I_{k+1}$, essentially agree on V_k; as x is a cluster point of $\langle x_n \rangle_{n \in I_{k+1}}$, we have $x(t) = \lim_{n \in I_{k+1}} x_n(t)$ for essentially all $t \in V_k$, and $x(t) = \lim_{n \in I} x_n(t)$ for essentially all $t \in V_k$. It follows at once that $\{t : x(t) \neq \lim_{n \in I} x_n(t)\}$ is meager, as required.

(b) For the general case, we consider the functions z_n on $X \times \mathbf{R}$, where

$$z_n(t,\alpha) = 1 \quad \text{if } \alpha \leqslant x_n(t), \qquad 0 \quad \text{otherwise.}$$

We have to observe first that $z_n \in \hat{\boldsymbol{B}}(X \times \mathbf{R})$; for if $Y \subseteq X$ is a dense G_δ set such that $x_n|_Y$ is continuous, then $Y \times \mathbf{R}$ is a dense G_δ set in $X \times \mathbf{R}$ such that $z_n|_{Y \times \mathbf{R}}$ is the characteristic function of a closed set. Next, $\{z_n : n \in \mathbf{N}\}$ is relatively countably compact in $\hat{\boldsymbol{B}}(X \times \mathbf{R})$; for if $I \subseteq \mathbf{N}$ is any infinite set, there is a non-principal ultrafilter $\mathcal{F}$ containing I such that $y = \lim_{n \to \mathcal{F}} x_n \in \hat{\boldsymbol{B}}(X)$; now, writing $w = \lim_{n \to \mathcal{F}} z_n$ in $\mathbf{R}^{X \times \mathbf{R}}$, we see that

$$\begin{aligned}\{(t,\alpha) : \alpha < y(t)\} &\subseteq \{(t,\alpha) : w(t,\alpha) = 1\} \\ &\subseteq \{(t,\alpha) : \alpha \leqslant y(t)\}.\end{aligned}$$

As the outer two sets have the Baire property, and their difference is meager, the inside set has the Baire property, and $w \in \hat{\boldsymbol{B}}(X \times \mathbf{R})$ is the required cluster point of $\langle z_n \rangle_{n \in I}$.

Now, we are given that x is a cluster point of $\langle x_n \rangle_{n \in \mathbf{N}}$ in $\mathbf{R}^X$; suppose that $\mathcal{F}$ is a non-principal ultrafilter on $\mathbf{N}$ such that $x = \lim_{n \to \mathcal{F}} x_n$. Then $z = \lim_{n \to \mathcal{F}} z_n$ is a cluster point of $\langle z_n \rangle_{n \in \mathbf{N}}$. So by part (a) we know that there is a subsequence $\langle z_n \rangle_{n \in I}$ such that $z(t,\alpha) = \lim_{n \in I} z_n(t,\alpha)$ for almost all (t,α). In this case, for any rationals $\alpha < \beta$,

$$\begin{aligned}&\left\{t : \limsup_{n \in I} x_n(t) > \beta > \alpha > x(t)\right\} \times [\alpha, \beta] \\ &\qquad \subseteq \left\{(t,\gamma) : \limsup_{n \in I} z_n(t,\gamma) = 1,\ z(t,\gamma) = 0\right\}\end{aligned}$$

is meager, so that $\{t : \limsup_{n \in I} x_n(t) > \beta > \alpha > x(t)\}$ is meager; as this is true whenever $\alpha < \beta$, $\{t : \limsup_{n \in I} x_n(t) > x(t)\}$ is meager. Similarly, $\{t : \liminf_{n \in I} x_n(t) < x(t)\}$ is meager, and $x(t) = \lim_{n \in I} x_n(t)$ for almost all t, as required.

6C. Corollary. *Let X be a Polish space, and $A \subseteq \hat{\boldsymbol{B}}(X)$ a countable, relatively countably compact set.* (a) *If $x \in \bar{A}$ in $\mathbf{R}^X$, then there is a sequence in A converging to x almost everywhere.* (b) *A is relatively compact in $\hat{\boldsymbol{B}}(X)$.*

Proof. Because A is countable, every point of $\bar{A}$ is a cluster point of a sequence in A, so that (a) is just a restatement of the theorem. Part (b) follows at once.

6D. Much of the interest of 6B, 6C above lies in their association with Theorem 2F of [6]. We give now two contrasting results, the first an example to show that the Lebesgue-measure version of 6C above is false, and the second a proof of a category version of Corollary 2I of [6].

Example. Let $X=[0,1]$, and for each $n \in \mathbf{N}$ let A_n be

$$\left\{\chi\left(\bigcup_{i \leqslant n} I_i\right): \text{each } I_i \text{ is a closed interval in } [0,1] \text{ with rational endpoints, } \sum_{i \leqslant n} \mu I_i \leqslant 2^{-n}\right\},$$

where μ is Lebesgue measure. It is easy to see that each A_n is relatively compact in $M_\mu(X)$. If we set $A=\bigcup_{n \in \mathbf{N}} A_n$, then A is countable, and every sequence in A either has a subsequence lying entirely in some A_n, which at once has a sub-subsequence converging everywhere, or has a subsequence meeting each A_n in at most one term, which will converge almost everywhere to 0. Thus A is relatively countably compact in the space $M_\mu(X)$. But A is actually dense in the set of all characteristic functions on X. So A is not relatively compact in $M_\mu(X)$.

Problem. It remains open, however, whether, if $A \subseteq M([0,1])$ is countable and relatively compact (rather than relatively countably compact), then every point of $\bar{A}$ is the limit almost everywhere of a sequence in A. (This is of course true subject to Axiom K of [6], by [6, Proposition 4I]. We have also been able to prove it subject to Martin's axiom.)

6E. Theorem. *Let X be a Polish space and $A \subseteq \hat{\boldsymbol{B}}(X)$ a countably compact set which is "separated" in the sense that if $x, y \in A$ and $x \neq y$, then $\{t: x(t) \neq y(t)\}$ is non-meager. Then A is metrizable for $\mathfrak{T}_p$.*

Proof. Since $\mathbf{R}$ is homeomorphic to $]0,1[$, there is no harm in supposing, as we shall henceforth, that $A \subseteq]0,1[^X$.

(a) Observe first that A is sequentially compact. For, by Theorem 6B, any sequence $\langle x_n \rangle_{n \in \mathbf{N}}$ in A has a subsequence $\langle x_n \rangle_{n \in I}$ converging almost everywhere. Now any two cluster points of $\langle x_n \rangle_{n \in I}$ in A are equal almost everywhere, and therefore equal, as A is separated. As A is countably compact, $\lim_{n \in I} x_n$ exists in A.

(b) It follows that if $\langle A_n \rangle_{n\in\mathbf{N}}$ is any decreasing sequence of sequentially closed sets in A and $z\in\bigcap_{n\in\mathbf{N}}\overline{A_n}$, the closures being taken in $\mathbf{R}^X$, then $z\in\overline{\bigcap_{n\in\mathbf{N}}A_n}$. For, given any neighborhood U of z in $\mathbf{R}^X$, we can choose $x_n\in A_n\cap U$ for each $n\in\mathbf{N}$. Now $\langle x_n\rangle_{n\in\mathbf{N}}$ has a convergent subsequence with a limit $x\in\overline{U}\cap\bigcap_{n\in\mathbf{N}}A_n$, as each A_n is sequentially closed. As $\mathfrak{T}_p$ is regular, this shows that $z\in\overline{\bigcap_{n\in\mathbf{N}}A_n}$.

(c) For the rest of this proof, we shall write $\boldsymbol{M}$ for the quotient of $\hat{\boldsymbol{B}}(X)$ by the subspace consisting of the functions zero almost everywhere. Then $\boldsymbol{M}$ is a Dedekind complete Riesz space with the countable sup property; moreover, there is a sequence $\langle u_n\rangle_{n\in\mathbf{N}}$ in $\boldsymbol{M}$ such that each $u_n>0$, and for every $u>0$ in $\boldsymbol{M}$, there is some $n\in\mathbf{N}$ such that $u\geqslant u_n$. (We pass over the case $X=\varnothing$.)

(d) Fix for the moment any $z\in\overline{A}$, taken in $\mathbf{R}^X$. We shall show that $z\in A$. We show first that there is a sequentially closed $A_1\subseteq A$ such that $z\in\overline{A_1}$ and whenever $C\subseteq A_1$ is sequentially closed and $z\in\overline{C}$, then $\sup_{x\in C}x^{\cdot}=\sup_{x\in A_1}x^{\cdot}$ in $\boldsymbol{M}$. For suppose, if possible, otherwise. Then we can choose sets C_ξ, for countable ordinals ξ, in such a way that

> $C_0=A$;
> given that C_ξ is sequentially closed in A and $z\in\overline{C_\xi}$, choose a sequentially closed $C_{\xi+1}\subseteq C_\xi$ such that $z\in\overline{C}_{\xi+1}$ and $v_{\xi+1}<v_\xi$, where $v_\xi=\sup\{x^{\cdot}:x\in C_\xi\}$;
> for countable limit ordinals $\xi>0$, set $C_\xi=\bigcap_{\eta<\xi}C_\eta$, and observe that $z\in\overline{C_\xi}$ by part (b).

But we now have a strictly decreasing family $\langle v_\xi\rangle_{\xi<\Omega}$ in $\boldsymbol{M}$, which is impossible because $\boldsymbol{M}$ has the countable sup property.

(e) Similarly, there is a sequentially closed $A_2\subseteq A_1$ such that $z\in\overline{A_2}$ and if $C\subseteq A_2$ is a sequentially closed set such that $z\in\overline{C}$, then $\inf_{x\in C}x^{\cdot}=w_0=\inf\{x^{\cdot}:x\in A_2\}$. Set $w_1=\sup\{x^{\cdot}:x\in A_2\}$.

Consider the sequence $\langle u_n\rangle_{n\in\mathbf{N}}$ of part (c). For each $n\in\mathbf{N}$,

$$D_n=\{x:x\in A_2,\ x^{\cdot}\leqslant w_1-u_n\}$$

is a sequentially closed subset of A_2. So its closure does not contain z, and there is a finite set $F_n\subseteq X$ such that no $x\in D_n$ agrees with z on F_n; i.e.,

$$x\in A_2,\quad x(t)=z(t)\ \ \forall t\in F_n\quad\Rightarrow\quad x^{\cdot}\nleqslant w_1-u_n.$$

Similarly, there is a finite $F_n'\subseteq X$ such that

$$x\in A_2,\quad x(t)=z(t)\ \ \forall t\in F_n'\quad\Rightarrow\quad x^{\cdot}\ngeqslant w_0+u_n.$$

If we set $F=\bigcup_{n\in\mathbf{N}}(F_n\cup F_n')$, then F is countable, and there is an $x\in A_2$ such that $x(t)=z(t)$ for every $t\in F$ (as A_2 is sequentially compact). But now we have

$$x^{\cdot}\leqslant w_1,\qquad x^{\cdot}\nleqslant w_1-u_n\quad\forall n\in\mathbf{N},$$

so that $x^{\cdot}=w_1$. In the same way, $x^{\cdot}=w_0$ and $w_0=w_1$. This shows that any two members of A_2 must be equal almost everywhere; as A is separated, A_2 must be a singleton—in fact, $A_2=\{z\}$, and $z\in A$.

(f) Thus we have shown that A is closed in $\mathbf{R}^X$, and therefore compact. We next suppose that $G\subseteq X$ is a non-empty open set and that $\alpha<\beta$ in $\mathbf{R}$. Then there is a finite $F=F(G,\alpha,\beta)\subseteq X$ such that

whenever $x,y\in A$ agree on F, then at least one of $\{t:t\in G,\ x(t)>\alpha\}$, $\{t:t\in G,\ x(t)<\beta\}$ is non-meager.

For set

$$A'=\{x:x\in A,\ x(t)\leqslant\alpha\text{ for almost all }t\in G\},$$
$$A''=\{x:x\in A,\ x(t)\geqslant\beta\text{ for almost all }t\in G\}.$$

Then both A' and A'' are sequentially closed in A, and therefore countably compact; by the argument of (a)–(e) above, they are compact. At the same time, because G is non-meager, they are disjoint. It follows at once that there must be some finite $F\subseteq X$ such that no $x\in A'$, $y\in A''$ can agree on F; i.e., if $x,y\in A$ agree on F, then either $x\notin A'$ or $y\notin A''$.

(g) If we now take a sequence $\langle V_n\rangle_{n\in\mathbf{N}}$ of non-empty open sets in X running over a base for the topology of X, then

$$F=\bigcup\{F(V_n,\alpha,\beta):n\in\mathbf{N},\ \alpha<\beta\text{ in }\mathbf{Q}\}$$

is countable. If $x,y\in A$ agree on F, then $x^{\cdot}=y^{\cdot}$ in M, so $x=y$. Thus the topology $\mathfrak{T}_p(A,F)$ of uniform convergence on finite subsets of F is a metrizable topology on A coarser than the compact topology $\mathfrak{T}_p(A,X)$. So the two are equal, and A is metrizable under $\mathfrak{T}_p$.

Remark. One of us (see [15]) has shown that, subject to Martin's axiom, the corresponding theorem is true in $M_\mu(X)$ for any measure μ of finite magnitude. We do not have a parallel category result for non-Polish spaces.

Addendum.

Note that versions of Lemma 2B and part of Theorem 2F may be found in [17]. In [18] we see that the definition of K-analytic space used in this paper is (for Hausdorff spaces) equivalent to those of Choquet [3] and Frolik. Concerning the remark of 4I(c), $[0,\Omega]$ can always be embedded in $B_3([0,1])$; see [16]. (We are indebted to W. Fleissner for the reference.)

A partial answer to the problem in 5K is the following: if the continuum hypothesis is true, then every bounded real-valued function on $[0,1]$ is in the closed convex hull of some compact set in $B_2([0,1])$. Thus Proposition 5D, as stated, may fail for functions f which are of the second Baire class, rather than the first, in the second variable. But the argument of Lemma 5C is plainly really about $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$-measurable functions; it is only by means of a trick that we can apply it in general. And if we take $\Sigma\hat{\otimes}_\sigma\mathfrak{B}$-measurable functions which are of known Baire class in the second variable, some remarkable results have recently been proved; see, for instance, [19].

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